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DEPARTMENT OF THE INTERIOR
BUREAU OF EDUCATION

BULLETIN, 1921, No. 32

THE
REORGANIZATION OF MATHEMATICS
IN SECONDARY EDUCATION

©

A SUMMARY OF THE REPORT BY
THE NATIONAL COMMITTEE ON
MATHEMATICAL REQUIREMENTS



WASHINGTON
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1922

THE
REORGANIZATION OF MATHEMATICS
IN SECONDARY EDUCATION

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THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS.

(Under the auspices of The Mathematical Association of America.)

OFFICERS.

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J. A. Foberg, vice chairman, State Department of Public Instruction, Harrisburg, Pa.

MEMBERS.

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E. H. Moore, University of Chicago.

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Representing the Association of Teachers of Mathematics in New England.²

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Representing the Association of Teachers of Mathematics in the Middle States and Maryland.

J. A. Foberg, director of mathematical instruction, State Department, Harrisburg, Pa.³

Representing the Central Association of Science and Mathematics Teachers.

A. C. Olney, commissioner of secondary education, Sacramento, Calif.

Raleigh Schorling, The Lincoln School, New York City.

P. H. Underwood, Ball High School, Galveston, Tex.

Eula A. Weeks, Cleveland High School, St. Louis, Mo.

¹ Prof. Moore took the place vacated in 1918 by the resignation of Oswald Veblen, Princeton University.

² Mr. Downey took the place vacated in 1919 by the resignation of G. W. Evans, Charlestown High School, Boston, Mass.

³ Until July, 1921, of the Crane Technical High School, Chicago, Ill.

INTRODUCTION.

The National Committee on Mathematical Requirements was organized in the late summer of 1916 under the auspices of the Mathematical Association of America for the purpose of giving national expression to the movement for reform in the teaching of mathematics, which had gained considerable headway in various parts of the country, but which lacked the power that coordination and united effort alone could give.

The original nucleus of the committee, appointed by Prof. E. R. Hedrick, then president of the association, consisted of the following: A. R. Crathorne, University of Illinois; E. H. Moore, University of Chicago; D. E. Smith, Columbia University; H. W. Tyler, Massachusetts Institute of Technology; Oswald Veblen, Princeton University; and J. W. Young, Dartmouth College, chairman. This committee was instructed to add to its membership so as to secure adequate representation of secondary school interests, and then to undertake a comprehensive study of the whole problem concerned with the improvement of mathematical education and to cover the field of secondary and collegiate mathematics.

This group held its first meeting in September, 1916, at Cambridge, Mass. At that meeting it was decided to ask each of the three large associations of secondary school teachers of mathematics (The Association of Teachers of Mathematics in New England, The Association of Teachers of Mathematics in the Middle States and Maryland, and the Central Association of Science and Mathematics Teachers) to appoint an official representative on the committee. At this time also a general plan for the work of the committee was outlined and agreed upon.

In response to the request above referred to the following were appointed by the respective associations: Miss Vevia Blair, Horace Mann School, New York, N. Y., representing the Middle States and Maryland association; G. W. Evans, Charlestown High School, Boston, Mass., representing the New England association;¹ and J. A. Foberg, Crane Technical High School, Chicago, Ill., representing the central association.

At later dates the following members were appointed: A. C. Olney, commissioner of secondary education, Sacramento, Calif.; Raleigh Schorling, The Lincoln School, New York City; P. H. Underwood, Ball High School, Galveston, Tex.; and Miss Eula A. Weeks, Cleveland High School, St. Louis, Mo.

From the very beginning of its deliberations the committee felt that the work assigned to it could not be done effectively without adequate financial support. The wide geographical distribution of its membership made a full attendance at meetings of the committee difficult if not impossible without financial resources sufficient to defray the traveling expenses of members, the expenses of clerical assistance, etc. Above all, it was felt that, in order to give to the ultimate recommendations of the committee the authority and effectiveness which they should have, it was necessary to arouse the interest and secure the active cooperation of teachers, administrators, and organizations throughout the country—that the work of the committee should represent a cooperative effort on a truly national scale.

For over two years, owing in large part to the World War, attempts to secure adequate financial support proved unsuccessful. Inevitably also the war interfered with the committee's work. Several members were engaged in war work² and the others were carrying extra burdens on account of such work carried on by their colleagues.

¹ Mr. Evans resigned in the summer of 1919, owing to an extended trip abroad; his place was taken by W. F. Downey, English High School, Boston, Mass.

² Prof. Veblen resigned in 1917 on account of the pressure of his war duties. His place was taken on the committee by Prof. C. N. Moore, University of Cincinnati.

In the spring of 1919, however, and again in 1920, the committee was fortunate in securing generous appropriations from the General Education Board of New York City for the prosecution of its work.²

This made it possible greatly to extend the committee's activities. The work was planned on a large scale for the purpose of organizing a truly nation-wide discussion of the problems facing the committee, and J. W. Young and J. A. Foberg were selected to devote their whole time to the work of the committee. Suitable office space was secured and adequate stenographic and clerical help was employed.

The results of the committee's work and deliberations are presented in the following report. A word as to the methods employed may, however, be of interest at this point. The committee attempted to establish working contact with all organizations of teachers and others interested in its problems and to secure their active assistance. Nearly 100 such organizations have taken part in this work. A list of these organizations will be found in the complete report of the committee. Provisional reports on various phases of the problem were submitted to these cooperating organizations in advance of publication, and criticisms, comments, and suggestions for improvement were invited from individuals and special cooperating committees. The reports previously published for the committee by the United States Bureau of Education³ and in *The Mathematics Teacher*⁴ and designated as "preliminary" are the result of this kind of cooperation. The value of such assistance can hardly be overestimated and the committee desires to express to all individuals, organizations, and educational journals that have taken part its hearty appreciation and thanks. The committee believes it is safe to say, in view of the methods used in formulating them, that the recommendations of this final report have the approval of the great majority of progressive teachers throughout the country.

No attempt has been made in this report to trace the origin and history of the various proposals and movements for reform nor to give credit either to individuals or organizations for initiating them. A convenient starting point for the history of the modern movement in this country may be found in E. H. Moore's presidential address before the American Mathematical Society in 1902.⁵ But the movement here is only one manifestation of a movement that is world-wide and in which very many individuals and organizations have played a prominent part. The student interested in this phase of the subject is referred to the extensive publications of the International Commission on the Teaching of Mathematics, to the Bibliography of the Teaching of Mathematics, 1900-1912, by D. E. Smith and C. Goldziher (U. S. Bur. of Educ., Bull., 1912, No. 29) and to the bibliography (since 1912) to be found in the complete report of the national committee (Ch. XVI).

The national committee expects to maintain its office, with a certain amount of clerical help, during the year 1921-22 and perhaps for a longer period. It is hoped that in this way it may continue to serve as a clearing house for all activities looking to the improvement of the teaching of mathematics in this country, and to assist in bringing about the effective adoption in practice of the recommendations made in the following report, with such modifications of them as continued study and experimentation may show to be desirable.

² Again in Nov., 1921, the General Education Board made appropriations to cover the expense of publishing and distributing the complete report of the committee and to enable the committee to carry on certain phases of its work during the year 1922.

³ The Reorganization of the First Courses in Secondary School Mathematics, U. S. Bureau of Education, Secondary School Circular, No. 5, February, 1920. 11 pp. Junior High School Mathematics, U. S. Bureau of Education, Secondary School Circular, No. 6, July, 1920. 10 pp. The Function Concept in Secondary School Mathematics, Secondary School Circular No. 8, June, 1921. 10 pp.

⁴ Terms and Symbols in Elementary Mathematics, *The Mathematics Teacher*, 14: 107-118, March, 1921. Elective Courses in Mathematics for Secondary Schools, *The Mathematics Teacher*, 14: 161-170, April, 1921. College Entrance Requirements in Mathematics, *The Mathematics Teacher*, 14: 224-245, May, 1921.

⁵ E. H. Moore: On the Foundations of Mathematics, *Bulletin of the American Mathematical Society*, vol. 9 (1902-3), p. 402; *Science*, 17: 401.

THE REORGANIZATION OF MATHEMATICS IN SECONDARY EDUCATION.

Chapter I.

A BRIEF OUTLINE OF THE REPORT.

The present chapter gives a brief general outline of the contents of this pamphlet for the purpose of orienting the reader and making it possible for him to gain quickly an understanding of its scope and the problems which it considers.

The valid aims and purposes of instruction in mathematics are considered in Chapter II. A formulation of such aims and a statement of general principles governing the committee's work is necessary as a basis for the later specific recommendations. Here will be found the reasons for including mathematics in the course of study for all secondary school pupils.

To the end that all pupils in the period of secondary education shall gain early a broad view of the whole field of elementary mathematics, and, in particular, in order to insure contact with this important element in secondary education on the part of the very large number of pupils who, for one reason or another, drop out of school by the end of the ninth year, the national committee recommends emphatically that the course of study in mathematics during the seventh, eighth, and ninth years contain the fundamental notions of arithmetic, of algebra, of intuitive geometry, of numerical trigonometry and at least an introduction to demonstrative geometry, and that this body of material be required of all secondary school pupils. A detailed account of this material is given in Chapter III. Careful study of the later years of our elementary schools, and comparison with European schools, have shown the vital need of reorganization of mathematical instruction, especially in the seventh and eighth years. The very strong tendency now evident to consider elementary education as ceasing at the end of the sixth school year, and to consider the years from the seventh to the twelfth inclusive as comprising years of secondary education, gives impetus to the movement for reform of the teaching of mathematics at this stage. The necessity for devising courses of study for the new junior high school, comprising the years seven, eight, and nine, enables us to

present a body of materials of instruction, and to propose organizations of this material that will be valid not only for junior high schools conducted as separate schools, but also for years seven and eight in the traditional eight-year elementary school and the first year of the four-year high school.

While Chapter III is devoted to a consideration of the body of material of instruction in mathematics that is regarded as of sufficient importance to form part of the course of study for all secondary school pupils, Chapter IV is devoted to consideration of the types of material that properly enter into courses of study for pupils who continue their study of mathematics beyond the minimum regarded as essential for all pupils. Here will be found recommendations concerning the traditional subject matter of the tenth, eleventh, and twelfth school years, and also certain material that heretofore has been looked upon in this country as belonging rather to college courses of study; as, for instance, the elementary ideas and processes of the calculus.

Chapter V is devoted to a study of the types of secondary school instruction in mathematics that may be looked upon as furnishing the best preparation for successful work in college. This study leads to the conclusion that there is no conflict between the needs of those pupils who ultimately go to college and those who do not. Certain very definite recommendations are made as to changes that appear desirable in the statement of college-entrance requirements and in the type of college-entrance examination.

Chapter VI contains lists of propositions and constructions in plane and in solid geometry. The propositions are classified in such a way as to separate from others of less importance those which are regarded as so fundamental that they should form the common minimum of any standard course in the subject. This chapter has close connection with the two chapters which immediately precede it.

The statement previously made in our preliminary reports and repeated in Chapter II, that the function concept should serve as a unifying element running throughout the instruction in mathematics of the secondary school, has brought many requests for a more precise definition of the rôle of the function concept in secondary school mathematics. Chapter VII is intended to meet this demand.

Recommendations as to the adoption and use of terms and symbols in elementary mathematics are contained in Chapter VIII. It is intended to present a norm embodying agreement as to best current practice. It will tend to restrict the irresponsible introduction of new terms and symbols, but it does not close the door entirely on innovations that may from time to time prove serviceable and desirable.

The chapters of the complete report thus far referred to appear in full in this summary. The remaining chapters of the complete report give for the most part the results of special investigations prepared for the national committee. The contents of these chapters are indicated sufficiently at the end of the present summary to enable the reader to decide whether or not he is interested in the studies mentioned, and whether or not he desires the complete report.

Copies of the complete report of the national committee, which will probably be ready for distribution in the spring of 1922, may be had, free of charge, upon application addressed to the chairman, Prof. J. W. Young, Hanover, N. H.

Chapter II.

AIMS OF MATHEMATICAL INSTRUCTION—GENERAL PRINCIPLES.

I. INTRODUCTION.

A discussion of mathematical education, and of ways and means of enhancing its value, must be approached first of all on the basis of a precise and comprehensive formulation of the valid aims and purposes of such education.¹ Only on such a basis can we approach intelligently the problems relating to the selection and organization of material, the methods of teaching and the point of view which should govern the instruction, and the qualifications and training of the teachers who impart it. Such aims and purposes of the teaching of mathematics, moreover, must be sought in the nature of the subject, the rôle it plays in the practical, intellectual, and spiritual life of the world, and in the interests and capacities of the students.

Before proceeding with the formulation of these aims, however, we may properly limit to some extent the field of our enquiry. We are concerned primarily with the period of secondary education—comprising, in the modern junior and senior high schools, the period beginning with the seventh and ending with the twelfth school year, and concerning itself with pupils ranging in age normally from 12 to 18 years. References to the mathematics of the grades below the seventh (mainly arithmetic) and beyond the senior high school will be only incidental.

Furthermore, we are primarily concerned at this point with what may be described as "general" aims, that is to say aims which are valid for large sections of the school population and which may properly be thought of as contributing to a general education as distinguished from the specific needs of vocational, technical, or professional education.

II. THE AIMS OF MATHEMATICAL INSTRUCTION.

With these limitations in mind we may now approach the problem of formulating the more important aims that the teaching of mathematics should serve. It has been customary to distinguish three

¹ Reference may here be made to the formulation of the principal aims in education to be found in the *Cardinal Principles of Secondary Education*, published by the U. S. Bureau of Education as Bulletin No. 55, 1918. The main objectives of education are there stated to be: 1. Health; 2. Command of fundamental processes; 3. Worthy home membership; 4. Vocation; 5. Citizenship; 6. Worthy use of leisure; 7. Ethical character. These objectives are held to apply to all education—elementary, secondary, and higher—and all subjects of instruction are to contribute to their achievement.

classes of aims: (1) Practical or utilitarian, (2) disciplinary, (3) cultural; and such a classification is indeed a convenient one. It should be kept clearly in mind, however, that the three classes mentioned are not mutually exclusive and that convenience of discussion rather than logical necessity often assigns a given aim to one or the other of the classes. Indeed any truly disciplinary aim is practical, and in a broad sense the same is true of cultural aims.

Practical aims.—By a practical or utilitarian aim, in the narrower sense, we mean then the immediate or direct usefulness in life of a fact, method or process in mathematics.

1. The immediate and undisputed utility of the fundamental processes of arithmetic in the life of every individual demands our first attention. The first instruction in these processes, it is true, falls outside the period of instruction which we are considering. By the end of the sixth grade the child should be able to carry out the four fundamental operations with integers and with common and decimal fractions accurately and with a fair degree of speed. This goal can be reached in all schools—as it is being reached in many—if the work is done under properly qualified teachers and if drill is confined to the simpler cases which alone are of importance in the practical life of the great majority. (See more specifically, Ch. III, pp. 7, 18.) Accuracy and facility in numerical computation are of such vital importance, however, to every individual that effective drill in this subject should be continued throughout the secondary school period, not in general as a separate topic, but in connection with the numerical problems arising in other work. In this numerical work, besides accuracy and speed, the following aims are of the greatest importance:

(a) A progressive increase in the pupil's understanding of the nature of the fundamental operations and power to apply them in new situations. The fundamental laws of algebra are a potent influence in this direction. (See 3, below.)

(b) Exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurements, the determination of the number of figures to be used in computing and to be retained in the result, and the like.

(c) The development of self-reliance in the handling of numerical problems, through the consistent use of checks on all numerical work.

2. Of almost equal importance to every educated person is an understanding of the language of algebra and the ability to use this language intelligently and readily in the expression of such simple quantitative relations as occur in every-day life and in the normal reading of the educated person.

Appreciation of the significance of formulas and ability to work out simple problems by setting up and solving the necessary equations

must nowadays be included among the minimum requirements of any program of universal education.

3. The development of the ability to understand and to use such elementary algebraic methods involves a study of the fundamental laws of algebra and at least a certain minimum of drill in algebraic technique, which, when properly taught, will furnish the foundation for an understanding of the significance of the processes of arithmetic already referred to. The essence of algebra as distinguished from arithmetic lies in the fact that algebra concerns itself with the operations upon numbers *in general*, while arithmetic confines itself to operations on *particular* numbers.

4. The ability to understand and interpret correctly graphical representations of various kinds, such as nowadays abound in popular discussions of current scientific, social, industrial, and political problems will also be recognized as one of the necessary aims in the education of every individual. This applies to the representation of statistical data, which is becoming increasingly important in the consideration of our daily problems, as well as to the representation and understanding of various sorts of dependence of one variable quantity upon another.

5. Finally, among the practical aims to be served by the study of mathematics should be listed familiarity with the geometric forms common in nature, industry, and life; the elementary properties and relations of these forms, including their mensuration; the development of space-perception; and the exercise of spatial imagination. This involves acquaintance with such fundamental ideas as congruence and similarity and with such fundamental facts as those concerning the sum of the angles of a triangle, the pythagorean proposition and the areas and volumes of the common geometric forms.

Among directly practical aims should also be included the acquisition of the ideas and concepts in terms of which the quantitative thinking of the world is done, and of ability to think clearly in terms of those concepts. It seems more convenient, however, to discuss this aim in connection with the disciplinary aims.

Disciplinary aims.—We would include here those aims which relate to mental training, as distinguished from the acquisition of certain specific skills discussed in the preceding section. Such training involves the development of certain more or less general characteristics and the formation of certain mental habits which, besides being directly applicable in the setting in which they are developed or formed, are expected to operate also in more or less closely related fields—that is, to “transfer” to other situations.

The subject of the transfer of training has for a number of years been a very controversial one. Only recently has there been any

evidence of agreement among the body of educational psychologists. We need not at this point go into detail as to the present status of disciplinary values since this forms the subject of a separate chapter in the complete report (Chap. IX; see also Chap. X). It is sufficient for our present purpose to call attention to the fact that most psychologists have abandoned two extreme positions as to transfer of training. The first asserted that a pupil trained to reason well in geometry would thereby be trained to reason equally well in any other subject; the second denied the possibility of any transfer, and hence the possibility of any general mental training. That the effects of training do transfer from one field of learning to another is now, however, recognized. The amount of transfer in any given case depends upon a number of conditions. If these conditions are favorable, there may be considerable transfer, but in any case the amount of transfer is difficult to measure. Training in connection with certain attitudes, ideals, and ideas is almost universally admitted by psychologists to have general value. It may, therefore, be said that, with proper restrictions, general mental discipline is a valid aim in education.

The aims which we are discussing are so important in the restricted domain of quantitative and spatial (i. e., mathematical or partly mathematical) thinking which every educated individual is called upon to perform that we do not need for the sake of our argument to raise the question as to the extent of transfer to less mathematical situations.

In formulating the disciplinary aims of the study of mathematics the following should be mentioned:

(1) The acquisition, in precise form, of those ideas or concepts in terms of which the quantitative thinking of the world is done. Among these ideas and concepts may be mentioned ratio and measurement (lengths, areas, volumes, weights, velocities, and rates in general, etc), proportionality and similarity, positive and negative numbers, and the dependence of one quantity upon another.

(2) The development of ability to think clearly in terms of such ideas and concepts. This ability involves training in—

(a) Analysis of a complex situation into simpler parts. This includes the recognition of essential factors and the rejection of the irrelevant.

(b) The recognition of logical relations between interdependent factors and the understanding and, if possible, the expression of such relations in precise form.

(c) Generalization; that is, the discovery, and formulation of a general law and an understanding of its properties and applications.

(3) The acquisition of mental habits and attitudes which will make the above training effective in the life of the individual. Among

such habitual reactions are the following: A seeking for relations and their precise expression; an attitude of enquiry; a desire to understand, to get to the bottom of a situation; concentration and persistence; a love for precision, accuracy, thoroughness, and clearness, and a distaste for vagueness and incompleteness; a desire for orderly and logical organization as an aid to understanding and memory.

(4) Many, if not all, of these disciplinary aims are included in the broad sense of the idea of relationship or dependence—in what the mathematician in his technical vocabulary refers to as a "function" of one or more variables. Training in "functional thinking," that is thinking in terms of relationships, is one of the most fundamental disciplinary aims of the teaching of mathematics.

Cultural aims.—By cultural aims we mean those somewhat less tangible but none the less real and important intellectual, ethical, esthetic or spiritual aims that are involved in the development of appreciation and insight and the formation of ideals of perfection. As will be at once apparent the realization of some of these aims must await the later stages of instruction, but some of them may and should operate at the very beginning.

More specifically we may mention the development or acquisition of—

(1) Appreciation of beauty in the geometrical forms of nature, art, and industry.

(2) Ideals of perfection as to logical structure: precision of statement and of thought; logical reasoning (as exemplified in the geometric demonstration); discrimination between the true and the false, etc.

(3) Appreciation of the power of mathematics—of what Byron expressively called "the power of thought, the magic of the mind"²—and the rôle that mathematics and abstract thinking, in general, has played in the development of civilization; in particular in science, in industry, and in philosophy. In this connection mention should be made of the religious effect, in the broad sense, which the study of the permanence of laws in mathematics and of the infinite tends to establish.³

III. THE POINT OF VIEW GOVERNING INSTRUCTION.

The practical aims enumerated above, in spite of their vital importance, may without danger be given a secondary position in seeking to formulate the general point of view which should govern the

² D. E. Smith: Mathematics in the Training for Citizenship, Teachers College Record, vol. 18, May, 1917, p. 6.

³ For an elaboration of the ideas here presented in the barest outline, the reader is referred to the article by D. E. Smith already mentioned and to his presidential address before the Mathematical Association of America, Wellesley, Mass., Sept. 7, 1921.

teacher, provided only that they receive due recognition in the selection of material and that the necessary minimum of technical drill is insisted upon.

The primary purposes of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

All topics, processes, and drill in technique which do not directly contribute to the development of the powers mentioned should be eliminated from the curriculum. It is recognized that in the earlier periods of instruction the strictly logical organization of subject matter⁴ is of less importance than the acquisition, on the part of the pupil, of experience as to facts and methods of attack on significant problems, of the power to see relations, and of training in accurate thinking in terms of such relations. Care must be taken, however, through the dominance of the course by certain general ideas that it does not become a collection of isolated and unrelated details.

Continued emphasis throughout the course must be placed on the development of ability to grasp and to utilize ideas, processes, and principles in the solution of concrete problems rather than on the acquisition of mere facility or skill in manipulation. The excessive emphasis now commonly placed on manipulation is one of the main obstacles to intelligent progress. On the side of algebra, the ability to understand its language and to use it intelligently, the ability to analyze a problem, to formulate it mathematically, and to interpret the result must be dominant aims. *Drill in algebraic manipulation should be limited to those processes and to the degree of complexity required for a thorough understanding of principles and for probable applications either in common life or in subsequent courses which a substantial proportion of the pupils will take.* It must be conceived throughout as a means to an end, not as an end in itself. Within these limits, skill in algebraic manipulation is important, and drill in this subject should be extended far enough to enable students to carry out the essential processes accurately and expeditiously.

On the side of geometry the formal demonstrative work should be preceded by a reasonable amount of informal work of an intuitive, experimental, and constructive character. Such work is of great value in itself; it is needed also to provide the necessary familiarity with geometric ideas, forms, and relations, on the basis of which

⁴ "The logical from the standpoint of subject matter represents the goal, the last term of training, not the point of departure." Dewey, "How We Think," p. 62.

alone intelligent appreciation of formal demonstrative work is possible.

The one great idea which is best adapted to unify the course is that of the *functional relation*. The concept of a variable and of the dependence of one variable upon another is of fundamental importance to everyone. It is true that the general and abstract form of these concepts can become significant to the pupil only as a result of very considerable mathematical experience and training. There is nothing in either concept, however, which prevents the presentation of specific concrete examples and illustrations of dependence even in the early parts of the course. Means to this end will be found in connection with the tabulation of data and the study of the formula and of the graph and of their uses.

The primary and underlying principle of the course should be the idea of relationship between variables, including the methods of determining and expressing such relationship. The teacher should have this idea constantly in mind, and the pupil's advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends. (For a more detailed discussion of these ideas see Chap. VII below.)

The general ideas which appear more explicitly in the course and under the dominance of one or another of which all topics should be brought are: (1) The formula, (2) graphic representation, (3) the equation, (4) measurement and computation, (5) congruence and similarity, (6) demonstration. These are considered in more detail in a later section of the report (Chaps. III and IV).

IV. THE ORGANIZATION OF SUBJECT MATTER.

"General" courses.—We have already called attention to the fact that, in the earlier periods of instruction especially, logical principles of organization are of less importance than psychological and pedagogical principles. In recent years there has developed among many progressive teachers a very significant movement away from the older rigid division into "subjects" such as arithmetic, algebra, and geometry, each of which shall be "completed" before another is begun, and toward a rational breaking down of the barriers separating these subjects, in the interest of an organization of subject matter that will offer a psychologically and pedagogically more effective approach to the study of mathematics.

There has thus developed the movement toward what are variously called "composite," "correlated," "unified," or "general" courses. The advocates of this new method of organization base their claims on the obvious and important interrelations between arithmetic, algebra, and geometry (mainly intuitive), which the student must grasp

before he can gain any real insight into mathematical methods and which are inevitably obscured by a strict adherence to the conception of separate "subjects." The movement has gained considerable new impetus by the growth of the junior high-school idea, and there can be little question that the results already achieved by those who are experimenting with the new methods of organization warrant the abandonment of the extreme "water-tight compartment" methods of presentation.

The newer method of organization enables the pupil to gain a broad view of the whole field of elementary mathematics early in his high-school course. In view of the very large number of pupils who drop out of school at the end of the eighth or the ninth school year or who for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older type of organization under which the pupil studied algebra alone during the ninth school year, to the complete exclusion of all contact with geometry.

It should be noted, however, that the specific recommendations as to content given in the next two chapters do not necessarily imply the adoption of a different type of organization of the materials of instruction. A large number of high schools will for some time continue to find it desirable to organize their courses of study in mathematics by subjects—algebra, plane geometry, etc. Such schools are urged to adopt the recommendations made with reference to the content of the separate subjects. These, in the main, constitute an essential simplification as compared with present practice. The economy of time that will result in courses in ninth-year algebra, for instance, will permit of the introduction of the newer type of material, including intuitive geometry and numerical trigonometry, and thus the way will be prepared for the gradual adoption in larger measure of the recommendations of this report.

At the present time it is not possible to designate any particular order of topics or any organization of the materials of instruction as being the best or as calculated most effectively to realize the aims and purposes here set forth. More extensive and careful experimental work must be done by teachers and administrators before any such designation can be made that shall avoid undesirable extremes and that shall bear the stamp of general approval. This experimental work will prove successful in proportion to the skill and insight exercised in adapting the aims and purposes of instruction to the interests and capacities of the pupils. One of the greatest weaknesses of the traditional courses is the fact that both the interests and the capacities of pupils have received insufficient consideration and study. For a detailed account of courses in mathematics at a num-

ber of the most successful experimental schools, the reader is referred to Chapter XII of the complete report.

Required courses.—The national committee believes that the material described in the next chapter should be required of all pupils, and that under favorable conditions this minimum of work can be completed by the end of the ninth school year. In the junior high school, comprising grades seven, eight, and nine, the course for these three years should be planned as a unit *with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years.* In particular, college-entrance requirements should, during these years, receive no specific consideration. Fortunately there appears to be no conflict of interest during this period between those pupils who ultimately go to college and those who do not; a course planned in accordance with the principle just enunciated will form a desirable foundation for college preparation. (See Ch. V.)

Similarly, in case of the at present more prevalent 8-4 school organization, the mathematical material of the seventh and eighth grades should be selected and organized as a unit with the same purpose; the same applies to the work of the first year (ninth grade) of the standard four-year high school, and to later years in which mathematics may be a required subject.

In the case of some elective courses the principle needs to be modified so as to meet whatever specific vocational or technical purposes the courses may have. (See Ch. IV.)

The movement toward correlation of the work in mathematics with other courses in the curriculum, notably those in science, is as yet in its infancy. The results of such efforts will be watched with the keenest interest.

The junior high-school movement.—Reference has several times been made to the junior high school. The national committee adopted the following resolution on April 24, 1920:

The national committee approves the junior high school form of organization, and urges its general adoption in the conviction that it will secure greater efficiency in the teaching of mathematics.

The committee on the reorganization of secondary education, appointed by the National Education Association, in its pamphlet on the "Cardinal Principles of Secondary School Education," issued in 1918 by the Bureau of Education, advocates an organization of the school system whereby the first six years shall be devoted to elementary education, and the following six years to secondary education to be divided into two periods which may be designated as junior and senior periods.

To those interested in the study of the questions relating to the history and present status of the junior high-school movement, the following books are recommended: *Principles of Secondary Education*, by Inglis, Houghton Mifflin & Co., 1918; *The Junior High School*, *The Fifteenth Yearbook (Pt. III)* of the National Society for the Study of Education, Public School Publishing Co., 1919; *The Junior High School*, by Bennett, Warwick & York, 1919; *The Junior High School*, by Briggs, Houghton Mifflin & Co., 1920; and *The Junior High School*, by Koos, Harcourt, Brace & Howe, 1920.

V. THE TRAINING OF TEACHERS.

While the greater part of this report concerns itself with the content of courses in mathematics, their organization and the point of view which should govern the instruction, and investigations relating thereto, the national committee must emphasize strongly its conviction that even more fundamental is the problem of the teacher—his qualifications and training, his personality, skill, and enthusiasm.

The greater part of the failure of mathematics is due to poor teaching. Good teachers have in the past succeeded, and continue to succeed, in achieving highly satisfactory results with the traditional material; poor teachers will not succeed even with the newer and better material.

The United States is far behind Europe in the scientific and professional training required of its secondary school teachers (see Ch. XIV of the complete report). The equivalent of two or three years of graduate and professional training in addition to a general college course is the normal requirement for secondary school teachers in most European countries. Moreover, the recognized position of the teacher in the community must be such as to attract men and women of the highest ability into the profession. This means not only higher salaries but smaller classes and more leisure for continued study and professional advancement. It will doubtless require a considerable time before the public can be educated to realize the wisdom of taxing itself sufficiently to bring about the desired result. But if this ideal is continually advanced and supported by sound argument there is every reason to hope that in time the goal may be reached.

In the meantime everything possible should be done to improve the present situation. One of the most vicious and widespread practices consists in assigning a class in mathematics to a teacher who has had no special training in the subject and whose interests lie elsewhere, because in the construction of the time schedule he or she happens to have a vacant period at the time. This is done on the principle, apparently, that "anybody can teach mathematics" by simply

following a textbook and devoting 90 per cent of the time to drill in algebraic manipulation or to reciting the memorized demonstration of a theorem in geometry.

It will be apparent from the study of this report that a successful teacher of mathematics must not only be highly trained in his subject and have a genuine enthusiasm for it but must have also peculiar attributes of personality and above all insight of a high order into the psychology of the learning process as related to the higher mental activities. Administrators should never lose sight of the fact that while mathematics if properly taught is one of the most important, interesting, and valuable subjects of the curriculum, it is also one of the most difficult to teach successfully.

Standards for teachers.—It is necessary at the outset to make a fundamental distinction between standards in the sense of requirements for appointment to teaching positions, and standards of scientific attainment which shall determine the curricula of colleges and normal schools aiming to give candidates the best practicable preparation. The former requirements should be high enough to insure competent teaching, but they must not be so high as to form a serious obstacle to admission to the profession even for candidates who have chosen it relatively late. The main factors determining the level of these requirements are the available facilities for preparation, the needs of the pupils, and the economic or salary conditions.

Relatively few young people deliberately choose before entering college the teaching of secondary mathematics as a life work. In the more frequent or more typical case the college student who will ultimately become a teacher of secondary mathematics makes the choice gradually, perhaps unconsciously, late in the college course or even after its completion, perhaps after some trial of teaching in other fields. The possible supply of young people who have the real desire to become teachers of mathematics is so meager in comparison with the almost unlimited needs of the country that every effort should be made to develop and maintain that desire and all possible encouragement given those who manifest it. If, as will usually be the case, the desire is associated with the necessary mathematical capacity, it will not be wise to hamper the candidate by requiring too high attainments, though as a matter of course he will need guidance in continuing his preparation for a profession of exceptional difficulty and exceptional opportunity.

Another factor which must tend to restrict requirements of high mathematical attainment is the importance to the candidate of breadth of preparation. In college he may be in doubt as to becoming a teacher of mathematics or physics or some other subject. It is unwise to hasten the choice. In many cases the secondary teacher

must be prepared in more than one field, and to the future teacher of mathematics preparation in physics and drawing, not to mention chemistry, engineering, etc., may be at least as valuable as purely mathematical college electives beyond the calculus.

In the second sense—of standards of scientific attainment to be held by the colleges and normal schools—these institutions should make every effort—

1. To awaken interest in the subject and the teaching of it in as many young people of the right sort as possible.
2. To give them the best possible opportunity for professional preparation and improvement, both before and after the beginning of teaching.

How the matter of requirements for appointment will actually work out in a given community will inevitably depend upon conditions of time and place, varying widely in character and degree. In many communities it is already practicable and customary to require not less than two years of college work in mathematics, including elementary calculus, with provision for additional electives. Such a requirement the committee would strongly recommend, recognizing, however, that in some localities it would be for the present too restrictive of the supply. In some cases preparation in the pedagogy, philosophy, and history of mathematics could be reasonably demanded or at least given weight; in other cases, any considerable time spent upon them would be of doubtful value. In all cases requirements should be carefully adjusted to local conditions with a view to recognizing the value both of broad and thorough training on the part of those entering the profession and of continued preparation by summer work and the like. Particular pains should be taken that such preparation is made accessible and attractive in the colleges and normal schools from which teachers are drawn.

It is naturally important that entrance to the profession should not be much delayed by needlessly high or extended requirements, and the danger of creating a teacher who may be too much a specialist for school work and too little for college training must be guarded against. There may naturally also be a wide difference between requirements in a strong school offering many electives and a weaker one or a junior high school. Practically, it may be fair to expect that the stronger schools will maintain their standards not by arbitrary or general requirements for entrance to the profession but often by recruiting from other schools teachers who have both high attainments and successful teaching experience.

Programs of courses for colleges and normal schools preparing teachers in secondary mathematics will be found in Chapter XIV of the complete report, together with an account of existing conditions.

Chapter III.

MATHEMATICS FOR YEARS SEVEN, EIGHT, AND NINE.

I. INTRODUCTION.

There is a well-marked tendency among school administrators to consider grades one to six, inclusive, as constituting the elementary school and to consider the secondary school period as commencing with the seventh grade and extending through the twelfth.¹ Conforming to this view, the contents of the courses of study in mathematics for grades seven, eight, and nine are considered together. In the succeeding chapter the content for grades 10, 11, and 12 is considered.

The committee is fully aware of the widespread desire on the part of teachers throughout the country for a detailed syllabus by years or half years which shall give the best order of topics with specific time allotments for each. This desire can not be met at the present time for the simple reason that no one knows what is the best order of topics nor how much time should be devoted to each in an ideal course. The committee feels that its recommendations should be so formulated as to give every encouragement to further experimentation rather than to restrict the teacher's freedom by a standardized syllabus.

However, certain suggestions as to desirable arrangements of the material are offered in a later section (Sec. III) of this chapter, and in Chapter XII (Mathematics in Experimental Schools) of the complete report there will be found detailed outlines giving the order of presentation and time allotments in actual operation in schools of various types. This material should be helpful to teachers and administrators in planning courses to fit their individual needs and conditions.

It is the opinion of the committee that the material included in this chapter should be required of all pupils. It includes mathematical knowledge and training which is likely to be needed by every citizen. Differentiation due to special needs should be made after and not before the completion of such a general minimum foundation. Such portions of the recommended content as have

¹ See Cardinal Principles of Secondary Education, p. 18.

"We therefore recommend a reorganization of the school system whereby the first six years shall be devoted to elementary education designed to meet the needs of pupils of approximately 6 to 12 years of age, and the second 6 years to secondary education designed to meet the needs of approximately 12 to 18 years of age. * * * The 6 years to be devoted to secondary education may well be divided into two periods which may be designated as the junior and senior periods."

not been completed by the end of the ninth year should be required in the following year.

The general principles which have governed the selection of the material presented in the next section and which should govern the point of view of the teaching have already been stated (Ch. II). At this point it seems desirable to recall specifically what was then said concerning principles governing the organization of material, the importance to be attached to the development of insight and understanding and of ability to think clearly in terms of relationships (dependence) and the limitations imposed on drill in algebraic manipulation. In addition we would call attention to the following:

It is assumed that at the end of the sixth school year the pupil will be able to perform with accuracy and with a fair degree of speed the fundamental operations with integers and with common and decimal fractions. The fractions here referred to are such simple ones in common use as are set forth in detail under A (c) in the following section. It may be pointed out that the standard of attainment here implied is met in a large number of schools, as is shown by various tests now in use (see Ch. XIII of the complete report), and can easily be met generally if time is not wasted on the relatively unimportant parts of the subject.

In adapting instruction in mathematics to the mental traits of pupils care should be taken to maintain the mental growth too often stunted by secondary school materials and methods, and an effort should be made to associate with inquisitiveness, the desire to experiment, the wish to know "how and why," and the like, the satisfaction of these needs.

In the years under consideration it is also especially important to give the pupils as broad an outlook over the various fields of mathematics as is consistent with sound scholarship. These years especially are the ones in which the pupil should have the opportunity to find himself, to test his abilities and aptitudes, and to secure information and experience which will help him choose wisely his later courses and ultimately his life work.

II. MATERIAL FOR GRADES SEVEN, EIGHT, AND NINE.

In the material outlined in the following pages no attempt is made to indicate the most desirable order of presentation. Stated by topics rather than years the mathematics of grades seven, eight, and nine may properly be expected to include the following:

A. Arithmetic:

- (a) The fundamental operations of arithmetic.
- (b) Tables of weights and measures in general practical use, including the most common metric units (meter, centimeter, millimeter, kilometer, gram, kilogram, liter). The meaning of such foreign monetary units as pound, franc, and mark.

- (c) Such simple fractions as $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{1}{6}$; others than these to have less attention.
- (d) Facility and accuracy in the four fundamental operations; time tests, taking care to avoid subordinating the teaching to the tests, or to use the tests as measures of the teacher's efficiency. (See Ch. XIII.)
- (e) Such simple short cuts in multiplication and division as that of replacing multiplication by 25 by multiplying by 100 and dividing by 4.
- (f) Percentage. Interchanging common fractions and per cents; finding any per cent of a number; finding what per cent one number is of another; finding a number when a cert. in per cent of it is known; and such applications of percentage as come within the student's experience.
- (g) Line, bar, and circle graphs wherever they can be used to advantage.
- (h) Arithmetic of the home: Household accounts, thrift, simple bookkeeping, methods of sending money, parcel post.
- Arithmetic of the community: Property and personal insurance, taxes.
- Arithmetic of banking: Savings accounts, checking accounts.
- Arithmetic of investment: Real estate, elementary notions of stocks and bonds, postal savings.
- (i) Statistics: Fundamental concepts, statistical tables and graphs; pictograms; graphs showing simple frequency distributions.

It will be seen that the material listed above includes some material of earlier instruction. This does not mean that this material is to be made the direct object of study but that drill in it shall be given in connection with the new work. It is felt that this shift in emphasis will make the arithmetic processes here involved much more effective and will also result in a great saving of time.

The amount of time devoted to arithmetic as a distinct subject should be greatly reduced from what is at present customary. This does not mean a lessening of emphasis on drill in arithmetic processes for the purpose of securing accuracy and speed. The need for continued arithmetic work and numerical computation throughout the secondary school period is recognized elsewhere in this report. (Ch. II.)

The applications of arithmetic to business should be continued late enough in the course to bring to their study the pupil's greatest maturity, experience, and mathematical knowledge, and to insure real significance of this study in the business and industrial life which many of the pupils will enter upon at the close of the eighth or ninth school year. (See I below.) In this connection care should be taken that the business practices taught in the schools are in accord with the best actual usage. Arithmetic should not be completed before the pupil has acquired the power of using algebra as an aid.

B. Intuitive geometry:

(a) The direct measurement of distances and angles by means of a linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(b) Areas of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of the corresponding formulas.

- (c) Practice in numerical computation with due regard to the number of figures used or retained.
- (d) Indirect measurement by means of drawings to scale. Uses of square ruled paper.
- (e) Geometry of appreciation. Geometric forms in nature, architecture, manufacture, and industry.
- (f) Simple geometric constructions with ruler and compasses, T-square, and triangle, such as that of the perpendicular bisector, the bisector of an angle, and parallel lines.
- (g) Familiarity with such forms as the equilateral triangle, the $30^{\circ}-60^{\circ}$ right triangle, and the isosceles right triangle; symmetry; a knowledge of such facts as those concerning the sum of the angles of a triangle and the Pythagorean relation; simple cases of geometric loci in the plane and in space.
- (h) Informal introduction to the idea of similarity.

The work in intuitive geometry should make the pupil familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to shape, size, and position. Much opportunity should be provided for exercising space perception and imagination. The simpler geometric ideas and relations in the plane may properly be extended to three dimensions. The work should, moreover, be carefully planned so as to bring out geometric relations and logical connections. Before the end of this intuitive work the pupil should have definitely begun to make inferences and to draw valid conclusions from the relations discovered. In other words, this informal work in geometry should be so organized as to make it a gradual approach to, and provide a foundation for, the subsequent work in demonstrative geometry.

C. Algebra:

1. The formula—its construction, meaning, and use (a) as a concise language; (b) as a shorthand rule for computation; (c) as a general solution; (d) as an expression of the dependence of one variable upon another.

The pupil will already have met the formula in connection with intuitive geometry. The work should now include translation from English into algebraic language, and vice versa, and special care should be taken to make sure that the new language is understood and used intelligently. The nature of the dependence of one variable in a formula upon another should be examined and analyzed, with a view to seeing "how the formula works." (See Ch. VII.)

2. Graphs and graphic representations in general—their construction and interpretation in (a) representing facts (statistical, etc.); (b) representing dependence; (c) solving problems.

After the necessary technique has been adequately presented graphic representation should not be considered as a separate topic.

but should be used throughout, whenever helpful, as an illustrative and interpretative instrument.

3. Positive and negative members—their meaning and use (*a*) as expressing both magnitude and one of two opposite directions or senses; (*b*) their graphic representation; (*c*) the fundamental operations applied to them.

4. The equation—its use in solving problems:

(*a*) Linear equations in one unknown--their solution and applications.

(*b*) Simple cases of quadratic equations when arising in connection with formulas and problems.

(*c*) Equations in two unknowns, with numerous concrete illustrations.

(*d*) Various simple applications of ratio and proportion in cases in which they are generally used in problems of similarity and in other problems of ordinary life. In view of the usefulness of the ideas and training involved, this subject may also properly include simple cases of variation.

5. Algebraic technique: (*a*) The fundamental operations.

Their connection with the rules of arithmetic should be clearly brought out and made to illuminate numerical processes. Drill in these operations should be limited strictly in accordance with the principle mentioned in Chapter II, page 9. In particular, "nests" of parentheses should be avoided, and multiplication and division should not involve much beyond monomial and binomial multipliers, divisors, and quotients.

(*b*) Factoring: The only cases that need be considered are (*i*) common factors of the terms of a polynomial; (*ii*) the difference of two squares; (*iii*) trinomials of the second degree that can be easily factored by trial.

(*c*) Fractions.

Here again the intimate connection with the corresponding processes of arithmetic should be made clear and should serve to illuminate such processes. The four fundamental operations with fractions should be considered only in connection with simple cases and should be applied constantly throughout the course so as to gain the necessary accuracy and facility.

(*d*) Exponents and radicals. The work done on exponents and radicals should be confined to the simplest material required for the treatment of formulas. The laws for positive integral exponents should be included. The consideration of radicals should be confined to transformations of the following types: $\sqrt{a^2b} = a\sqrt{b}$, $\sqrt{a/b} = \frac{1}{b}\sqrt{ab}$ and $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$, and to the numerical evaluation of simple expressions involving the radical sign. A process for finding the square

root of a number should be included, but not for finding the square root of a polynomial.

(e) Stress should be laid upon the need for checking solutions.

D. Numerical trigonometry:

(a) Definition of sine, cosine, and tangent.

(b) Their elementary properties as functions.

(c) Their use in solving problems involving right triangles.

(d) The use of tables of these functions (to three or four places).

The introduction of the elementary notions of trigonometry into the earlier courses in mathematics has not been as general in the United States as in foreign countries. (See Ch. XI of the complete report.) Among the reasons for early introduction of this topic are these: Its practical usefulness for many citizens; the insight it gives into the nature of mathematical methods, particularly those concerned with indirect measurement, and into the rôle that mathematics plays in the life of the world; the fact that it is not difficult and that it offers wide opportunity for concrete and significant application, and the interest it arouses in the pupils. It should be based upon the work in intuitive geometry, with which it has intimate contacts (see *B*, *d*, *h*), and should be confined to the simplest material needed for the numerical treatment of the problems indicated. Relations between the trigonometric functions need not be considered.

E. Demonstrative geometry.—The demonstration of a limited number of propositions, with no attempt to limit the number of fundamental assumptions, the principal purpose being to show to the pupil what "demonstration" means.

Many of the geometric facts previously inferred intuitively may be used as the basis upon which the demonstrative work is built. This is not intended to preclude the possibility of giving at a later time rigorous proofs of some of the facts inferred intuitively. It should be noted that from the strictly logical point of view the attempt to reduce to a minimum the list of axioms, postulates or assumptions is not at all necessary, and from a pedagogical point of view such an attempt in an elementary course is very undesirable. It is necessary, however, that those propositions which are to be used as the basis of subsequent formal proofs be explicitly listed and their logical significance recognized.

In regard to demonstrative geometry some teachers have objected to the introduction of such work below the tenth grade on the ground that with such immature pupils as are found in the ninth grade nothing worth while could be accomplished in the limited time available. These teachers may be right with regard to conditions prevailing or likely to prevail in the majority of schools in the immediate future. The committee has therefore in a later section of this

chapter (Sec. III) made alternative provision for the omission of work in demonstrative geometry.

On the other hand, it is proper to call attention to the fact that certain teachers have successfully introduced a limited amount of work in demonstrative geometry into the ninth grade (see Ch. XII of the complete report), and that it would seem desirable that others should make the experiment when conditions are favorable. Much of the opposition is probably due to a failure to realize the extent to which the work in intuitive geometry, if properly organized, will prepare the way for the more formal treatment, and to a misconception of the purposes and extent of the work in demonstrative geometry that is proposed. In reaching a decision on this question teachers should keep in mind that it is one of their important duties and obligations, in the grades under consideration, to show their pupils the nature, content, and possibilities of later courses in their subject and to give to each pupil an opportunity to determine his aptitudes and preferences therefor. The omission in the earlier courses of all work of a demonstrative nature in geometry would disregard one educationally important aspect of mathematics.

F. History and biography.—Teachers are advised to make themselves reasonably acquainted with the leading events in the history of mathematics, and thus to know that mathematics has developed in answer to human needs, intellectual as well as technical. They should use this material incidentally throughout their courses for the purpose of adding to the interest of the pupils by means of informal talks on the growth of mathematics and on the lives of the great makers of the science.

G. Optional topics.—Certain schools have been able to cover satisfactorily the work suggested in sections A-F before the end of the ninth grade. (See Ch. XII, on Experimental Schools.) The committee looks with favor on the efforts, in such schools, to introduce earlier than is now customary certain topics and processes which are closely related to modern needs, such as the meaning and use of fractional and negative exponents, the use of the slide rule, the use of logarithms and of other simple tables, and simple work in arithmetic and geometric progressions, with modern applications to such financial topics as interest and annuities and to such scientific topics as falling bodies and laws of growth.

H. Topics to be omitted or postponed.—In addition to the large amount of drill in algebraic technique already referred to, the following topics should, in accordance with our basic principles, be excluded from the work of grades seven, eight, and nine; some of them will properly be included in later courses (see Ch. IV):

Highest common factor and lowest common multiple, except the simplest cases involved in the addition of simple fractions.

The theorems on proportion relating to alternation, inversion, composition, and division.

Literal equations, except such as appear in common formulas, including the derivation of formulas and of geometric relations, or to show how needless computation may be avoided.

Radicals, except as indicated in a previous section.

Square root of polynomials.

Cube root.

Theory of exponents.

Simultaneous equations in more than two unknowns.

The binomial theorem.

Imaginary and complex numbers.

Radical equations except such as arise in dealing with elementary formulas.

I. Problems.—As already indicated, much of the emphasis now generally placed on the formal exercise should be shifted to the “concrete” or “verbal” problem. The selection of problem material is, therefore, of the highest importance.

The demand for “practical” problems should be fully met in so far as the maturity and previous experience of the pupil will permit. But above all, the problems must be “real” *to the pupil*, must connect with his ordinary thought, and must be within the world of his experience and interest.

The educational utility of problems is not to be measured by their commercial or scientific value, but by their degree of reality for the pupils. They must exemplify those leading ideas which it is desired to impart, and they must do so through media which are real to those under instruction. The reality is found in the students, the utility in their acquisition of principles.²

There should be, moreover, a conscious effort through the selection of problems to correlate the work in mathematics with the other courses of the curriculum, especially in connection with courses in science. The introduction of courses in “general science” increases the opportunities in this direction.

J. Numerical computation, use of tables, etc.—The solution of problems should offer opportunity throughout the grades under consideration for considerable arithmetical and computational work. In this connection attention should be called to the importance of exercising common sense and judgment in the use of approximate data, keeping in mind the fact that all data secured from measurement are approximate. A pupil should be led to see the absurdity of giving the area of a circle to a thousandth of a square inch when the radius has been measured only to the nearest inch. He should understand the conception of “the number of significant figures” and should not retain more figures in his result than are warranted by the accuracy of his data. The ideals of accuracy and of self-reliance and the necessity of checking all numerical results should be emphasized. An insight into the nature of tables, including some elementary notions as to interpolation, is highly desirable. The use of tables of various

²Carson: Mathematical Education, pp. 42-45.

kinds (such as squares and square roots, interest and trigonometric functions) to facilitate computation and to develop the idea of dependence should be encouraged.

III. SUGGESTED ARRANGEMENTS OF MATERIAL.

In approaching the problem of arranging or organizing this material it is necessary to consider the different situations that may have to be met.

1. The junior high school.--In view of the fact that under this form of school organization pupils may be expected to remain in school until the end of the junior high-school period instead of leaving in large numbers at the end of the eighth school year, the mathematics of the three years of the junior high school should be planned as a unit, and should include the material recommended in the preceding section. There remains the question as to the order in which the various topics should be presented and the amount of time to be devoted to each. The committee has already stated its reasons for not attempting to answer this question (see Sec. I). The following plans for the distribution of time are, however, suggested in the hope that they may be helpful, but no one of them is recommended as superior to the others, and only the large divisions of material are mentioned.

PLAN A.

First year: Applications of arithmetic, particularly in such lines as relate to the home, to thrift, and to the various school subjects; intuitive geometry.

Second year: Algebra; applied arithmetic, particularly in such lines as relate to the commercial, industrial and social needs.

Third year: Algebra, trigonometry, demonstrative geometry.

By this plan the demonstrative geometry is introduced in the third year, and arithmetic is practically completed in the second year.

PLAN B.

First year: Applied arithmetic (as in plan A); intuitive geometry.

Second year: Algebra, intuitive geometry, trigonometry.

Third year: Applied arithmetic, algebra, trigonometry, demonstrative geometry.
By this plan trigonometry is taken up in two years, and the arithmetic is transferred from the second year to the third year.

PLAN C.

First year: Applied arithmetic (as in plan A), intuitive geometry, algebra.

Second year: Algebra, intuitive geometry.

Third year: Trigonometry, demonstrative geometry, applied arithmetic.

By this plan algebra is confined chiefly to the first two years.

PLAN D.

First year: Applied arithmetic (as in plan A), intuitive geometry.

Second year: Intuitive geometry, algebra.

Third year: Algebra, trigonometry, applied arithmetic.

By this plan demonstrative geometry is omitted entirely.

PLAN E.

First year: Intuitive geometry, simple formulas, elementary principles of statistics, arithmetic (as in plan A).

Second year: Intuitive geometry, algebra, arithmetic.

Third year: Geometry, numerical trigonometry, arithmetic.

2. Schools organized on the 8-4 plan.—It can not be too strongly emphasized that, in the case of the older and at present more prevalent plan of the 8-4 school organization, the work in mathematics of the seventh, eighth, and ninth grades should also be organized to include the material here suggested.

The prevailing practice of devoting the seventh and eighth grades almost exclusively to the study of arithmetic is generally recognized as a wasteful marking of time. It is mainly in these years that American children fall behind their European brothers and sisters. No essentially new arithmetical principles are taught in these years, and the attempt to apply the previously learned principles to new situations in the more advanced business and economic aspects of arithmetic is doomed to failure on account of the fact that the situations in question are not and can not be made real and significant to pupils of this age. We need only refer to what has already been said in this chapter on the subject of problems.

The same principles should govern the selection and arrangement of material in mathematics for the seventh and eighth grades of a grade school as govern the selection for the corresponding grades of a junior high school, with this exception: Under the 8-4 form of organization many pupils will leave school at the end of the eighth year. This fact must receive due consideration. The work of the seventh and eighth years should be so planned as to give the pupils in these grades the most valuable mathematical information and training that they are capable of receiving in those years, with little reference to courses that they may take in later years. As to possibilities for arrangement, reference may be made to the plans given above for the first two years of the junior high school. When the work in mathematics of the seventh and eighth grades has been thus reorganized, the work of the first year of a standard four-year high school should complete the program suggested.

Finally, there must be considered the situation in those four-year high schools in which the pupils have not had the benefit of the reorganized instruction recommended for grades seven and eight. It may be hoped that this situation will be only temporary, although it must be recognized, that owing to a variety of possible reasons (lack of adequately prepared teachers in grades seven and eight, lack of suitable text books, administrative inertia, and the like), the new plans will not be immediately adopted and that therefore, for some years, many high schools will have to face the situation implied.

In planning the work of the ninth grade under these conditions teachers and administrative officers should again be guided by the principle of giving the pupils the most valuable mathematical information and training which they are capable of receiving in this year with little reference to future courses which the pupil may or may

not take. It is to be assumed that the work of this year is to be required of all pupils. Since for many this will constitute the last of their mathematical instruction, it should be so planned as to give them the widest outlook consistent with sound scholarship.

Under these conditions it would seem desirable that the work of the ninth grade should contain both algebra and geometry. It is, therefore, recommended that about two-thirds of the time be devoted to the most useful parts of algebra, including the work on numerical trigonometry, and that about one-third of the time be devoted to geometry, including the necessary informal introduction and, if feasible, the first part of demonstrative geometry.

It should be clear that owing to the greater maturity of the pupils much less time need be devoted in the ninth grade to certain topics of intuitive geometry (such as direct measurement, for example) than is desirable when dealing with children in earlier grades. Even under the conditions presupposed pupils will be acquainted with most of the fundamental geometric forms and with the mensuration of the most important plane and solid figures. The work in geometry in the ninth grade can then properly be made to center about indirect measurement and the idea of similarity (leading to the processes of numerical trigonometry), and such geometric relations as the sum of the angles of a triangle, the Pythagorean proposition, congruence of triangles, parallel and perpendicular lines, quadrilaterals and the more important simple constructions.

Chapter IV.

MATHEMATICS FOR YEARS TEN, ELEVEN, AND TWELVE.

I. INTRODUCTION.

The committee has in the preceding chapter expressed its judgment that the material there recommended for the seventh, eighth, and ninth years should be required of all pupils. In the tenth, eleventh, and twelfth years, however, the extent to which elections of subjects is permitted will depend on so many factors of a general character that it seems unnecessary and inexpedient for the present committee to urge a positive requirement beyond the minimum one already referred to. The subject must, like others, stand or fall on its intrinsic merit or on the estimate of such merit by the authorities responsible at a given time and place. The committee believes nevertheless that every standard high school should not merely offer courses in mathematics for the tenth, eleventh, and twelfth years, but should encourage a large proportion of its pupils to take them. Apart from the intrinsic interest and great educational value of the study of mathematics, it will in general be necessary for those preparing to enter college or to engage in the numerous occupations involving the use of mathematics to extend their work beyond the minimum requirement.

The present chapter is intended to suggest for students in general courses the most valuable mathematical training that will appropriately follow the courses outlined in the previous chapter. Under present conditions most of this work will normally fall in the last three years of the high school; that is, in general, in the tenth, eleventh, and twelfth years.

The selection of material is based on the general principles formulated in Chapter II. At this point attention need be directed only to the following:

1. In the years under consideration it is proper that some attention be paid to the students' vocational or other later educational needs.
2. The material for these years should include as far as possible those mathematical ideas and processes that have the most important applications in the modern world. As a result, certain material will naturally be included that at present is not ordinarily given in secondary-school courses; as, for instance, the material concerning the calculus. On the other hand, certain other material that is now

included in college entrance requirements will be excluded. The results of an investigation made by the national committee in connection with a study of these requirements indicates that modifications to meet these changes will be desirable from the standpoint of both college and secondary school (see Ch. V).

3. During the years now under consideration an increasing amount of attention should be paid to the logical organization of the material, with the purpose of developing habits of logical memory, appreciation of logical structure, and ability to organize material effectively.

It can not be too strongly emphasized that the broadening of content of high-school courses in mathematics suggested in the present and in previous chapters will materially increase the usefulness of these courses to those who pursue them. It is of prime importance that educational administrators and others charged with the advising of students should take careful account of this fact in estimating the relative importance of mathematical courses and their alternatives. The number of important applications of mathematics in the activities of the world is to-day very large and is increasing at a very rapid rate. This aspect of the progress of civilization has been noted by all observers who have combined a knowledge of mathematics with an alert interest in the newer developments in other fields. It was revealed in very illuminating fashion during the recent war by the insistent demand for persons with varying degrees of mathematical training for many war activities of the first moment. If the same effort were made in time of peace to secure the highest level of efficiency available for the specific tasks of modern life, the demand for those trained in mathematics would be no less insistent; for it is in no wise true that the applications of mathematics in modern warfare are relatively more important or more numerous than its applications in those fields of human endeavor which are of a constructive nature.

There is another important point to be kept in mind in considering the relative value to the average student of mathematical and various alternative courses. If the student who omits the mathematical courses has need of them later, it is almost invariably more difficult, and it is frequently impossible, for him to obtain the training in which he is deficient. In the case of a considerable number of alternative subjects a proper amount of reading in spare hours at a more mature age will ordinarily furnish him the approximate equivalent to that which he would have obtained in the way of information in a high-school course in the same subject. It is not, however, possible to make up deficiencies in mathematical training in so simple a fashion. It requires systematic work under a competent teacher to master properly the technique of the subject, and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the indi-

vidual discovers his need for further mathematical training it is usually difficult for him to take the time from his other activities for systematic work in elementary mathematics.

II. RECOMMENDATIONS FOR ELECTIVE COURSES.

The following topics are recommended for inclusion in the mathematical electives open to pupils who have satisfactorily completed the work outlined in the preceding chapter, comprising arithmetic, the elementary notions of algebra, intuitive geometry, numerical trigonometry, and a brief introduction to demonstrative geometry.

1. *Plane demonstrative geometry.*—The principal purposes of the instruction in this subject are: To exercise further the spatial imagination of the student, to make him familiar with the great basal propositions and their applications, to develop understanding and appreciation of a deductive proof and the ability to use this method of reasoning where it is applicable and to form habits of precise and succinct statement, of the logical organization of ideas, and of logical memory. Enough time should be spent on this subject to accomplish these purposes.

The following is a suggested list of topics under which the work in demonstrative geometry may be organized:¹ (a) Congruent triangles, perpendicular bisectors, bisectors of angles; (b) arcs, angles, and chords in circles; (c) parallel lines and related angles, parallelograms; (d) the sum of the angles for triangle and polygon; (e) secants and tangents to circles with related angles, regular polygons; (f) similar triangles, similar figures; (g) areas; numerical computation of lengths and areas, based upon geometric theorems already established.

Under these topics constructions, loci, areas, and other exercises are to be included.

It is recommended that the formal theory of limits and of incommensurable cases be omitted, but that the ideas of limit and of incommensurable magnitudes receive informal treatment.

It is believed that a more frequent use of the idea of motion in the demonstration of theorems is desirable, both from the point of view of gaining greater insight and of saving time.²

If the great basal theorems are selected and effectively organized into a logical system, a considerable reduction (from 30 to 40 per cent) can be made in the number of theorems given either in the Harvard list or in the report of the Committee of Fifteen. Such a reduction is exhibited in the lists prepared by the committee and

¹ It is not intended that the order here given should imply anything as to the order of presentation. (See also Ch. VI.)

² Reference may here be made to the treatment given in recent French texts such as those by Bourlet and Méray.

printed later in this report (Ch. VI). In this connection it may be suggested that more attention than is now customary may profitably be given to those methods of treatment which make consistent use of the idea of motion (already referred to), continuity (the tangent as the limit of a secant, etc.), symmetry, and the dependence of one geometric magnitude upon another.

If the student has had a satisfactory course in intuitive geometry and some work in demonstration before the tenth grade, he may find it possible to cover a minimum course in demonstrative geometry, giving the great basal theorems and constructions, together with exercises, in the 90 periods constituting a half year's work.

2. *Algebra.* —(a) Simple functions of one variable: Numerous illustrations and problems involving linear, quadratic, and other simple functions including formulas from science and common life. More difficult problems in variation than those included in the earlier course.

(b) Equations in one unknown: Various methods for solving a quadratic equation (such as factoring, completing the square, use of formula) should be given. In connection with the treatment of the quadratic a very brief discussion of complex numbers should be included. Simple cases of the graphic solution of equations of degree higher than the second should be discussed and applied.

(c) Equations in two or three unknowns: The algebraic solution of linear equation in two or three unknowns and the graphic solution of linear equations in two unknowns should be given. The graphic and algebraic solution of a linear and a quadratic equation and of two quadratics that contain no first degree term and no xy term should be included.

(d) Exponents, radicals and logarithms: The definitions of negative, zero and fractional exponents should be given, and it should be made clear that these definitions must be adopted if we wish such exponents to conform to the laws for positive integral exponents. Reduction of radical expressions to those involving fractional exponents should be given as well as the inverse transformation. The rules for performing the fundamental operations on expressions involving radicals, and such transformations as

$$\sqrt[n]{a} \sqrt[n]{b} = \frac{1}{b} \sqrt[n]{ab^{n-1}}, \quad \sqrt[n]{a^n b} = a \sqrt[n]{b}, \quad \frac{a}{\sqrt[n]{b} - \sqrt[n]{c}} = \frac{a(\sqrt[n]{b} - \sqrt[n]{c})}{b - c}$$

should be included. In close connection with the work on exponents and radicals there should be given as much of the theory of logarithms as is involved in their application to computation and sufficient practice in their use in computation to impart a fair degree of facility.

(e) Arithmetic and geometric progressions: The formulas for the n th term and the sum of n terms should be derived and applied to significant problems.

(f) Binomial theorem: A proof for positive integral exponents should be given; it may also be stated that the formula applies to the case of negative and fractional exponents under suitable restrictions, and the problems may include the use of the formula in these cases as well as in the case of positive integral exponents.

3. *Solid geometry*.—The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them. It is felt that the work in plane geometry gives enough training in logical demonstration to warrant a shifting of emphasis in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility in visualizing spatial relations and figures, in representing such figures on paper, and in solving problems in mensuration.

For many of the practical applications of mathematics it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the mathematical courses, preferably not later than the beginning of the eleventh school year. Some schools will find it possible and desirable to introduce the more elementary notions of solid geometry in connection with related ideas of plane geometry.

The work in solid geometry should include numerous exercises in computation based on the formulas established. This will serve to correlate the work with arithmetic and algebra and to furnish practice in computation.

The following provisional outline of subject matter is submitted:

- a. Propositions relating to lines and planes, and to dihedral and trihedral angles.
- b. Mensuration of the prism, pyramid, and frustum; the (right circular) cylinder, cone and frustum, based on an informal treatment of limits; the sphere, and the spherical triangle.
- c. Spherical geometry.
- d. Similar solids.

Such theorems as are necessary as a basis for the topics here outlined should be studied in immediate connection with them.

Desirable simplification and generalization may be introduced into the treatment of mensuration theorems by employing such theorems as Cavalieri's and Simpson's, and the Prismoid Formula; but rigorous proofs or derivations of these need not be included.

Beyond the range of the mensuration topics indicated above, it seems preferable to employ the methods of the elementary calculus. (See section 6, below).

It should be possible to complete a minimum course covering the topics outlined above in not more than one-third of a year.

The list of propositions in solid geometry given in Chapter VI should be considered in connection with the general principles stated at the beginning of this section. By requiring formal proofs to a more limited extent than has been customary, time will be gained to attain the aims indicated and to extend the range of geometrical information of the pupil. Care must be exercised to make sure that the pupil is thoroughly familiar with the facts, with the associated terminology, with all the necessary formulas, and that he secures the necessary practice in working with and applying the information acquired to concrete problems.

4. *Trigonometry*.—The work in elementary trigonometry begun in the earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions and their use in proving identities and in solving easy trigonometric equations. The use of the transit in connection with the simpler operations of surveying and of the sextant for some of the simpler astronomical observations, such as those involved in finding local time, is of value; but when no transit or sextant is available, simple apparatus for measuring angles roughly may and should be improvised. Drawings to scale should form an essential part of the numerical work in trigonometry. The use of the slide rule in computations requiring only three-place accuracy and in checking other computations is also recommended.

5. *Elementary statistics*.—Continuation of the earlier work to include the meaning and use of fundamental concepts and simple frequency distributions with graphic representations of various kinds and measures of central tendency (average, mode, and median).

6. *Elementary calculus*.—The work should include:

(a) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

(b) Applications of derivatives to easy problems in rates and in maxima and minima.

(c) Simple cases of inverse problems; e. g., finding distance from velocity, etc.

(d) Approximate methods of summation leading up to integration as a powerful method of summation.

(e) Applications to simple cases of motion, area, volume, and pressure.

Work in the calculus should be largely graphic and may be closely related to that in physics; the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that, while the elementary calculus is sufficiently easy, interesting, and valuable to justify its introduction, special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry. No possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers or all pupils in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for calculus later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool. The applications of elementary calculus to simple concrete problems are far more abundant and more interesting than those of algebra. The necessary technique is extremely simple. The subject is commonly taught in secondary schools in England, France, and Germany, and appropriate English texts are available.³

7. *History and biography.*—Historical and biographical material should be used throughout to make the work more interesting and significant.

8. *Additional electives.*—Additional electives such as *mathematics of investment, shop mathematics, surveying and navigation, descriptive or projective geometry* will appropriately be offered by schools which have special needs or conditions, but it seems unwise for the national committee to attempt to define them pending the results of further experience on the part of these schools.

III. PLANS FOR ARRANGEMENT OF THE MATERIAL.

In the majority of high schools at the present time the topics suggested can probably be given most advantageously as separate units of a three-year program. However, the national committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present much of this material more effectively in combined courses unified by one or more of such central ideas, functionality and graphic representation.

³ Quotations and typical problems from one of these texts will be found in a supplementary note appended to this chapter.

As to the arrangement of the material the committee gives below four plans which may be suggestive and helpful to teachers in arranging their courses. No one of them is, however, recommended as superior to the others.

PLAN A.

Tenth year: Plane demonstrative geometry, algebra.
Eleventh year: Statistics, trigonometry, solid geometry.
Twelfth year: The calculus, other elective.

PLAN B.

Tenth year: Plane demonstrative geometry, solid geometry.
Eleventh year: Algebra, trigonometry, statistics.
Twelfth year: The calculus, other elective.

PLAN C.

Tenth year: Plane demonstrative geometry, trigonometry.
Eleventh year: Solid geometry, algebra, statistics.
Twelfth year: The calculus, other elective.

PLAN D.

Tenth year: Algebra, statistics, trigonometry.
Eleventh year: Plane and solid geometry.
Twelfth year: The calculus, other elective.

Additional information on ways of organizing this material will be found in Chapter XII on Mathematics in Experimental Schools.

SUPPLEMENTARY NOTE ON THE CALCULUS AS A HIGH-SCHOOL SUBJECT.

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high-school program? How can it be expected that high-school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the committee in recommending the inclusion of elementary work in the calculus in the high-school program.

By the calculus we mean for the present purpose a study of *rates of change*. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions can not be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future many secondary-school teachers will have any occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the committee the introduction of elementary calculus necessarily includes sufficient

algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynomial is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations—and such training is of value to everyone.

The following typical extracts from an English text intended for use in secondary schools may be quoted:

"It has been said that the calculus is that branch of mathematics which schoolboys understand and senior wranglers fail to comprehend. * * * So long as the graphic treatment and practical applications of the calculus are kept in view, the subject is an extremely easy and attractive one. Boys can be taught the subject early in their mathematical career, and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. * * * The phenomena must first be known practically and then studied philosophically. To reverse the order of these processes is impossible."

The text in question, after an interesting historical sketch, deals with such problems as the following:

A train is going at the rate of 40 miles an hour. Represent this graphically.

At what rate is the length of the daylight increasing or decreasing on December 31, March 26, etc.? (From tabular data.)

A cart going at the rate of 5 miles per hour passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone. Find when and where the bicycle will overtake the cart.

A man has 4 miles of fencing wire and wishes to fence in a rectangular piece of prairie land through which a straight river flows, the bank of the stream being utilized as one side of the inclosure. How can he do this so as to inclose as much land as possible?

A circular tin canister closed at both ends has a surface area of 100 square centimeters. Find the greatest volume it can contain.

Post-office regulations prescribe that the combined length and girth of a parcel must not exceed 6 feet. Find the maximum volume of a parcel whose shape is a prism with the ends square.

A pulley is fixed 15 feet above the ground, over which passes a rope 30 feet long with one end attached to a weight which can hang freely, and the other end is held by a man at a height of 3 feet from the ground. The man walks horizontally away from beneath the pulley at the rate of 3 feet per second. Find the rate at which the weight rises when it is 10 feet above the ground.

The pressure on the surface of a lake due to the atmosphere is known to be 14 pounds per square inch. The pressure in the liquid x inches below the surface is known to be given by the law $dp/dx=0.036$. Find the pressure in the liquid at a depth of 10 feet.

The arch of a bridge is parabolic in form. It is 5 feet wide at the base and 5 feet high. Find the volume of water that passes through per second in a flood when the water is rushing at the rate of 10 feet per second.

A force of 20 tons compresses the spring buffer of a railway stop through 1 inch, and the force is always proportional to the compression produced. Find the work done by a train which compresses a pair of such stops through 6 inches.

These may illustrate the aims and point of view of the proposed work. It will be noted that not all of them involve calculus, but those that do not lead up to it.

Chapter V.

COLLEGE ENTRANCE REQUIREMENTS.

The present chapter is concerned with a study of topics and training in elementary mathematics that will have most value as preparation for college work, and with recommendations of definitions of college-entrance requirements in elementary algebra and plane geometry.

General considerations.—The primary purpose of college-entrance requirements is to test the candidate's ability to benefit by college instruction. This ability depends, so far as our present inquiry is concerned, upon (1) general intelligence, intellectual maturity and mental power; (2) specific knowledge and training required as preparation for the various courses of the college curriculum.

Mathematical ability appears to be a sufficient but not a necessary condition for general intelligence.¹ For this, as well as for other reasons, it would appear that *college-entrance requirements in mathematics should be formulated primarily on the basis of the special knowledge and training required for the successful study of courses which the student will take in college.*

The separation of prospective college students from the others in the early years of the secondary school is neither feasible nor desirable. It is therefore obvious that secondary-school courses in mathematics can not be planned with specific reference to college-entrance requirements. Fortunately there appears to be no real conflict of interest between those students who ultimately go to college and those who do not, so far as mathematics is concerned. It will be made clear in what follows that a course in this subject, covering from two to two and one-half years in a standard four-year high school, and so planned as to give the most valuable mathematical training which the student is capable of receiving, will provide adequate preparation for college work.

Topics to be included in high-school courses.—In the selection of material of instruction for high-school courses in mathematics, its value as preparation for college courses in mathematics need not be specifically considered. Not all college students study mathematics; it is therefore reasonable to expect college departments in this sub-

¹ A recent investigation made by the department of psychology at Dartmouth College showed that all students of high rank in mathematics had a high rating on general intelligence; the converse was not true, however.

ject to adjust themselves to the previous preparation of their students. Nearly all college students do, however, study one or more of the physical sciences (astronomy, physics, chemistry) and one or more of the social sciences (history, economics, political science, sociology). Entrance requirements must therefore insure adequate mathematical preparation in these subjects. Moreover, it may be assumed that adequate preparation for these two groups of subjects will be sufficient for all other subjects for which the secondary schools may be expected to furnish the mathematical prerequisites.

The national committee recently conducted an investigation for the purpose of securing information as to the content of high-school courses of instruction most desirable from the point of view of preparation for college work. A number of college teachers, prominent in their respective fields, were asked to assign to each of the topics in the following table an estimate of its value as preparation for the elementary courses in their respective subjects. Table I gives a summary of the replies, arranged in two groups—"Physical sciences," including astronomy, physics, and chemistry; and "Social sciences," including history, economics, sociology, and political science.

The high value attached to the following topics is significant: Simple formulas—their meaning and use; the linear and quadratic functions and variation; numerical trigonometry; the use of logarithms and other topics relating to numerical computation; statistics. These all stand well above such standard requirements as arithmetic and geometric progression, binomial theorem, theory of exponents, simultaneous equations involving one or two quadratic equations, and literal equations.

These results would seem to indicate that a modification of present college-entrance requirements in mathematics is desirable from the point of view of college teachers in departments other than mathematics. It is interesting to note how closely the modifications suggested by this inquiry correspond to the modifications in secondary-school mathematics foreshadowed by the study of needs of the high-school pupil irrespective of his possible future college attendance. The recommendations made in Chapter II that functional relationship be made the "underlying principle of the course," that the meaning and use of simple formulas be emphasized, that more attention be given to numerical computation (especially to the methods relating to approximate data), and that work on numerical trigonometry and statistics be included, have received widespread approval throughout the country. That they should be in such close accord with the desires of college teachers in the fields of physical and social sciences as to entrance requirements is striking. We find here the justification for the belief expressed earlier in this report that there is no real conflict between the needs of students who ultimately go to college and those who do not.

TABLE 1.—*Value of topics as preparation for elementary college courses.*

[In the headings of the table, E—essential, C—of considerable value, S—of some value, O—of little or no value, N—number of replies received. The figures in the first four columns of each group are percentages of the number of replies received.]

| | Physical sciences. | | | | | Social sciences. | | | | |
|--|--------------------|----|----|----|----|------------------|----|----|----|----|
| | E. | C. | S. | O. | N. | E. | C. | S. | O. | N. |
| Negative numbers—their meaning and use..... | 79 | 5 | 10 | 5 | 39 | 45 | 17 | 22 | 17 | 18 |
| Imaginary numbers—their meaning and use..... | 23 | 21 | 25 | 31 | 39 | 13 | 13 | 37 | 37 | 16 |
| Simple fractions—their meaning and use..... | 93 | 5 | 2 | — | 41 | 47 | 26 | 21 | 5 | 19 |
| Graphical representation of statistical data..... | 57 | 25 | 15 | 3 | 40 | 37 | 24 | 14 | 5 | 21 |
| Graphs (mathematical and empirical): | | | | | | | | | | |
| As a method of representing dependence..... | 62 | 16 | 22 | — | 37 | 15 | 54 | 15 | 15 | 13 |
| As a method of solving problems..... | 45 | 20 | 28 | 6 | 25 | 18 | 46 | 18 | 11 | 11 |
| The linear function, $y = mx + b$ | 78 | 14 | 8 | — | 37 | 29 | 29 | 13 | 29 | 14 |
| The quadratic function, $y = ax^2 + bx + c$ | 59 | 21 | 17 | 3 | 34 | 8 | 8 | 33 | 50 | 12 |
| Equations: Problems leading to— | | | | | | | | | | |
| Linear equations in one unknown..... | 98 | 2 | — | — | 41 | 40 | 7 | 20 | 33 | 15 |
| Quadratic equations in one unknown..... | 78 | 15 | 5 | 2 | 40 | 31 | 8 | 8 | 54 | 13 |
| Simultaneous linear equations in 2 unknowns..... | 71 | 24 | 3 | 3 | 38 | 33 | — | — | 58 | 12 |
| Simultaneous linear equations in more than 2 unknowns..... | 43 | 29 | 23 | 6 | 35 | 8 | 8 | 17 | 67 | 12 |
| One quadratic and one linear equation in 2 unknowns..... | 40 | 24 | 27 | 9 | 33 | — | 9 | 9 | 82 | 11 |
| Two quadratic equations in 2 unknowns..... | 31 | 19 | 28 | 22 | 32 | — | 9 | — | 91 | 11 |
| Equations of higher degree than the second..... | 10 | 32 | 32 | 26 | 31 | — | — | 9 | 91 | 11 |
| LITERAL equations (other than formulas)..... | 43 | 18 | 32 | 7 | 28 | — | 10 | 40 | 50 | 10 |
| Ratio and proportion..... | 84 | 8 | 3 | 5 | 39 | 37 | 26 | 32 | 5 | 19 |
| Variation..... | 50 | 13 | 20 | 17 | 30 | 17 | 33 | 25 | 25 | 12 |
| Numerical computation: | | | | | | | | | | |
| With approximate data—rational use of significant figures..... | 61 | 36 | — | 3 | 39 | 40 | 27 | 20 | 13 | 12 |
| Short-cut methods..... | 27 | 38 | 24 | 10 | 37 | 29 | 35 | 23 | 12 | 17 |
| Use of logarithms..... | 62 | 29 | 7 | 2 | 42 | 12 | 29 | 29 | 29 | 17 |
| Use of other tables to facilitate computation..... | 24 | 45 | 26 | 5 | 38 | 18 | 20 | 41 | 12 | 17 |
| Use of slide rule..... | 24 | 39 | 26 | 12 | 38 | 11 | 39 | 28 | 22 | 18 |
| Theory of exponents..... | 36 | 31 | 25 | 8 | 36 | — | 21 | 21 | 57 | 14 |
| Theory of logarithms..... | 34 | 26 | 21 | 18 | 38 | 7 | 13 | 20 | 60 | 15 |
| Arithmetic progression..... | 16 | 32 | 38 | 13 | 37 | 23 | 20 | 12 | 35 | 17 |
| Geometric progression..... | 19 | 27 | 40 | 14 | 37 | 23 | 25 | 18 | 35 | 17 |
| Binomial theorem..... | 35 | 32 | 18 | 13 | 37 | 13 | 20 | 27 | 40 | 15 |
| Probability..... | 9 | 32 | 41 | 19 | 32 | 20 | 35 | 35 | 10 | 20 |
| Statistics: | | | | | | | | | | |
| Meaning and use of elementary concepts..... | 23 | 28 | 31 | 17 | 29 | 55 | 36 | 5 | 5 | 22 |
| Frequency distributions and frequency curves..... | 15 | 19 | 35 | 32 | 26 | 47 | 33 | 10 | 10 | 21 |
| Correlation..... | 11 | 18 | 39 | 32 | 28 | 33 | 47 | 14 | 5 | 21 |
| Numerical trigonometry: | | | | | | | | | | |
| Use of sine, cosine, and tangent in the solution of simple problems involving right triangles..... | 68 | 21 | 3 | 8 | 38 | — | — | 25 | 75 | 12 |
| Demonstrative geometry..... | 68 | 15 | 12 | 6 | 34 | — | 21 | 43 | 36 | 14 |
| Plane trigonometry (usual course)..... | 57 | 27 | 11 | 5 | 37 | 8 | 23 | 31 | 38 | 13 |
| Analytic geometry: | | | | | | | | | | |
| Fundamental conceptions and methods in the plane..... | 32 | 45 | 19 | 3 | 31 | — | 15 | 38 | 46 | 13 |
| Systematic treatment of— | | | | | | | | | | |
| Straight line..... | 34 | 37 | 20 | 9 | 35 | 9 | 9 | 18 | 64 | 11 |
| Circle..... | 29 | 43 | 20 | 9 | 35 | — | 18 | 9 | 73 | 11 |
| Conic sections..... | 18 | 41 | 26 | 15 | 34 | — | 9 | 18 | 73 | 11 |
| Polar coordinates..... | 18 | 26 | 41 | 15 | 34 | — | — | 18 | 82 | 11 |
| Empirical curves and fitting curves to observations..... | 12 | 38 | 38 | 12 | 34 | 8 | — | 25 | 67 | 12 |

TABLE 2.—Topics in order of value as preparation for elementary college courses.

[The figures in the column headed "E" are taken from Table 1, taking in each case the higher of the two "E" ratings there given. The column headed "E+C" gives in each case the sum of the two ratings for "E" and "C." An asterisk indicates that the topic in question is now included in the definitions of the college entrance examination board.]

| | E. | E+C. |
|---|----|------|
| *Linear equations in one unknown..... | 98 | 100 |
| Simple formulas—their meaning and use..... | 93 | 98 |
| *Ratio and proportion..... | 81 | 92 |
| *Negative numbers—their meaning and use..... | 79 | 84 |
| *Quadratic equations in one unknown..... | 78 | 93 |
| The linear function: $y=mx+b$ | 78 | 92 |
| *Simultaneous linear equations in two unknowns..... | 71 | 95 |
| Numerical trigonometry—the use of the sine, cosine, and tangent in the solution of simple problems involving right triangles..... | 68 | 89 |
| *Elementary geometry..... | 68 | 83 |
| Use of logarithms in computation..... | 62 | 91 |
| *Graphs as a method of representing dependence..... | 62 | 78 |
| Computation with approximate data—rational use of significant figures..... | 61 | 97 |
| The quadratic function: $y=ax^2+bx+c$ | 59 | 89 |
| Plane trigonometry—usual course..... | 57 | 84 |
| Graphic representation of statistical data..... | 57 | 82 |
| Statistics—meaning and use of elementary concepts..... | 55 | 91 |
| Variation..... | 50 | 63 |
| Statistics—frequency distributions and curves..... | 47 | 80 |
| *Graphic solution of problems..... | 45 | 65 |
| *Literal equations..... | 43 | 61 |
| *Simultaneous linear equations in more than 2 unknowns..... | 43 | 72 |
| *Simultaneous equations, one quadratic, one linear..... | 40 | 61 |
| *Theory of exponents..... | 36 | 67 |
| *Binomial theorem..... | 35 | 67 |
| Analytic geometry of the straight line..... | 34 | 71 |
| Theory of logarithms..... | 31 | 60 |
| Statistics—correlation..... | 33 | 80 |
| Analytic geometry—fundamental conceptions..... | 32 | 77 |
| *Simultaneous quadratic equations..... | 31 | 50 |
| Analytic geometry of the circle..... | 29 | 72 |
| Short-cut methods of computation..... | 29 | 65 |
| Use of tables in computation (other than logarithms)..... | 24 | 69 |
| Use of slide rule. ² | 24 | 63 |
| Imaginary numbers..... | 24 | 44 |
| *Arithmetic progression..... | 23 | 52 |
| *Geometric progression..... | 23 | 48 |
| Probability..... | 20 | 55 |
| Conic sections..... | 18 | 59 |
| Polar coordinates..... | 18 | 44 |
| Empirical curves and fitting curves to observations..... | 12 | 50 |
| Equations of higher degree than the second..... | 10 | 42 |

² The list includes all the requirements of the college entrance examination board except those relating to algebraic technique. The topic of "Negative numbers" has also been given an asterisk, as it is clearly implied, though not explicitly mentioned, in the C. E. E. B. definitions.

The attitude of the colleges.—Mathematical instruction in this country is at present in a period of transition. While a considerable number of our most progressive schools have for several years given courses embodying most of the recommendations contained in Chapters II, III, and IV of the present report, the large majority of schools are still continuing the older types of courses or are only just beginning to introduce modifications. The movement toward reorganization is strong, however, throughout the country, not only in the standard four-year high schools but also in the newer junior high schools.

During this period of transition it should be the policy of the colleges, while exerting a desirable steady influence, to help the movement toward a sane reorganization. In particular they should take care not to place obstacles in the way of changes which are

clearly in the interest of more effective college preparation, as well as of better general education.

College-entrance requirements will continue to exert a powerful influence on secondary-school teaching. Unless they reflect the spirit of sound progressive tendencies, they will constitute a serious obstacle.

In the present chapter revised definitions of college-entrance requirements in plane geometry and elementary algebra are presented. So far as plane geometry is concerned, the problem of definition is comparatively simple. The proposed definition of the requirement in plane geometry does not differ from the one now in effect under the college entrance examination board. A list of propositions and constructions has however been prepared, and is given in the next chapter for the guidance of teachers and examiners.

In elementary algebra a certain amount of flexibility is obviously necessary both on account of the quantitative differences among colleges and of the special conditions attending a period of transition. The former differences are recognized by the proposal of a minor and a major requirement in elementary algebra. The second of these includes the first and is intended to correspond with the two-unit rating of the C. E. E. B.

In connection with this matter of units, the committee wishes particularly to disclaim any emphasis upon a special number of years or hours. The unit terminology is doubtless too well established to be entirely ignored in formulating college-entrance requirements, but the standard definition of unit³ has never been precise, and will now become much less so with the inclusion of the newer six-year program. A time allotment of 4 or 5 hours per week in the seventh year can certainly not have the same weight as the same number of hours in the twelfth year, and the disparity will vary with different subjects. *What is really important is the amount of subject matter and the quality of work done in it.* The "unit" can not be anything but a crude approximation to this. The distribution of time in the school program should not be determined by any arbitrary unit scale.

As a further means of securing reasonable flexibility, the committee recommends that for a limited time—say five years—the option be offered between examinations based on the old and on the new definitions, so far as differences between them may make this desirable.

In view of the changes taking place at the present time in mathematical courses in secondary schools, and the fact that college-entrance

The following definition, formulated by the National Committee on Standards of Colleges and Secondary Schools, has been given the approval of the C. E. E. B. "A unit represents a year's study in any subject in a secondary school, constituting approximately a quarter of a full year's work. A four-year secondary school curriculum should be regarded as representing not more than 16 units of work."

requirements should so soon as possible reflect desirable changes and assist in their adoption, the national committee recommends that either the American Mathematical Society or the Mathematical Association of America (or both) maintain a permanent committee on college-entrance requirements in mathematics, such a committee to work in close cooperation with other agencies which are now or may in the future be concerned in a responsible way with the relations between colleges and secondary schools.

DEFINITION OF COLLEGE ENTRANCE REQUIREMENTS.

ELEMENTARY ALGEBRA.

Minor requirement.—The meaning, use, and evaluation (including the necessary transformations) of simple formulas involving ideas with which the student is familiar and the derivation of such formulas from rules expressed in words.

The dependence of one variable upon another. Numerous illustrations and problems involving the linear function $y = mx + b$. Illustrations and problems involving the quadratic function $y = kx^2$.

The graph and graphic representations in general; their construction and interpretation, including the representation of statistical data and the use of the graph to exhibit dependence.

Positive and negative numbers; their meaning and use.

Linear equations in one unknown quantity; their use in solving problems.

Sets of linear equations involving two unknown quantities; their use in solving problems.

Ratio, as a case of simple fractions; proportion without the theorems on alternation, etc.; and simple cases of variation.

The essentials of algebraic technique. This should include—

(a) The four fundamental operations.

(b) Factoring of the following types: Common factors of the terms of a polynomial; the difference of two squares; trinomials of the second degree (including the square of a binomial) that can be easily factored by trial.

(c) Fractions, including complex fractions of a simple type.

(d) Exponents and radicals. The laws for positive integral exponents; the meaning and use of fractional exponents, but not the formal theory. The consideration of radicals may be confined to the simplification of expressions of the form $\sqrt{a^2b}$ and $\sqrt{a/b}$ and to the evaluation of simple expressions involving the radical sign. A process for extracting the square root of a number should be included but not the process for extracting the square root of a polynomial.

Numerical trigonometry. The use of the sine, cosine, and tangent in solving right triangles. The use of three or four place tables of natural functions.

Major requirement.—In addition to the minor requirement as specified above, the following should be included:

Illustrations and problems involving the quadratic function $y = ax^2 + bx + c$.

Quadratic equations in one unknown; their use in solving problems.

Exponents and radicals. Zero and negative exponents, and more extended treatment of fractional exponents. Rationalizing denominators. Solution of simple types of radical equations.

The use of logarithmic tables in computation without the formal theory.

Elementary statistics, including a knowledge of the fundamental concepts and simple frequency distributions, with graphic representations of various kinds.

The binomial theorem for positive integral exponents less than 8; with such applications as compound interest.

The formula for the n th term, and the sum of n terms, of arithmetic and geometric progressions, with applications.

Simultaneous linear equations in three unknown quantities and simple cases of simultaneous equations involving one or two quadratic equations; their use in solving problems.

Drill in algebraic manipulation should be limited, particularly in the minor requirement, by the purpose of securing a thorough understanding of important principles and facility in carrying out those processes which are fundamental and of frequent occurrence either in common life or in the subsequent courses that a substantial proportion of the pupils will study. Skill in manipulation must be conceived of throughout as a means to an end, not as an end in itself. Within these limits, skill and accuracy in algebraic technique are of prime importance, and drill in this subject should be extended far enough to enable students to carry out the fundamentally essential processes accurately and with reasonable speed.

The consideration of literal equations, when they serve a significant purpose, such as the transformation of formulas, the derivation of a general solution (as of the quadratic equation), or the proof of a theorem, is important. As a means for drill in algebraic technique they should be used sparingly.

The solution of problems should offer opportunity throughout the course for considerable arithmetical and computational work. The conception of algebra as an extension of arithmetic should be made significant both in numerical applications and in elucidating algebraic

principles. Emphasis should be placed upon the use of common sense and judgment in computing from approximate data, especially with regard to the number of figures retained, and on the necessity for checking the results. The use of tables to facilitate computation (such as tables of squares and square roots, of interest, and of trigonometric functions) should be encouraged.

PLANE GEOMETRY.

The usual theorems and constructions of good textbooks, including the general properties of plane rectilinear figures; the circle and the measurement of angles; similar polygons; areas; regular polygons and the measurement of the circle. The solution of numerous original exercises, including locus problems. Applications to the mensuration of lines and plane surfaces.

The scope of the required work in plane geometry is indicated by the List of Fundamental Propositions and Constructions, which is given in the next chapter. This list indicates in Section I the type of proposition which, in the opinion of the committee, may be assumed without proof or given informal treatment. Section II contains 52 propositions and 19 constructions which are regarded as so fundamental that they should constitute the common minimum of all standard courses in plane geometry. Section III gives a list of subsidiary theorems which suggests the type of additional propositions that should be included in such courses.

College-entrance examinations.—College-entrance examinations exert in many schools, and especially throughout the eastern section of the country, an influence on secondary school teaching which is very far-reaching. It is, therefore, well within the province of the national committee to inquire whether the prevailing type of examination in mathematics serves the best interests of mathematical education and of college preparation.

The reason for the almost controlling influence of entrance examinations in the schools referred to is readily recognized. Schools sending students to such colleges for men as Harvard, Yale, and Princeton, to the larger colleges for women, or to any institution where examinations form the only or prevailing mode of admission, inevitably direct their instruction toward the entrance examination. This remains true even if only a small percentage of the class intends to take these examinations, the point being that the success of a teacher is often measured by the success of his or her students in these examinations.

In the judgment of the committee, the prevailing type of entrance examination in algebra is primarily a test of the candidate's skill in

formal manipulation, and not an adequate test of his understanding or of his ability to apply the principles of the subject. Moreover, it is quite generally felt that the difficulty and complexity of the formal manipulative questions, which have appeared on recent papers set by colleges and by such agencies as the College Entrance Examination Board, has often been excessive. As a result, teachers preparing pupils for these examinations have inevitably been led to devote an excessive amount of time to drill in algebraic technique, without insuring an adequate understanding of the principles involved. Far from providing the desired facility, this practice has tended to impair it. For "practical skill, modes of effective technique, can be intelligently, nonmechanically used only when intelligence has played a part in their acquisition." (*Dewey, How We Think*, p. 52.)

Moreover, it must be noted that authors and publishers of textbooks are under strong pressure to make their content and distribution of emphasis conform to the prevailing type of entrance examination. Teachers in turn are too often unable to rise above the textbook. An improvement in the examinations in this respect will cause a corresponding improvement in textbooks and in teaching.

On the other hand, the makers of entrance examinations in algebra cannot be held solely responsible for the condition described. Theirs is a most difficult problem. Not only can they reply that as long as algebra is taught as it is, examinations must be largely on technique,⁴ but they can also claim with considerable force that technical facility is the only phase of algebra that can be fairly tested by an examination; that a candidate can rarely do himself justice amid unfamiliar surroundings and subject to a time limit on questions involving real thinking in applying principles to concrete situations; and that we must face here a real limitation on the power of an examination to test attainment. Many, and perhaps most, teachers will agree with this claim. Past experience is on their side; no generally accepted and effective "power test" in mathematics has as yet been devised and, if devised, it might not be suitable for use under conditions prevailing during an entrance examination.

But if it is true that the power of an examination is thus inevitably limited, the wisdom and fairness of using it as the sole means of admission to college is surely open to grave doubt. That many unqualified candidates are admitted under this system is not open to question. Is it not probable that many qualified candidates are at the same time excluded? If the entrance examination is a fair test of manipulative skill only, should not the colleges use additional means for learning something about the candidate's other abilities and qualifications?

⁴ The vicious circle is now complete. Algebra is taught mechanically because of the character of the entrance examination; the examination, in order to be fair, must conform to the character of the teaching.

Some teachers believe that an effective "power test" in mathematics is possible. Efforts to devise such a test should receive every encouragement.

In the meantime, certain desirable modifications of the prevailing type of entrance examinations are possible. The college entrance examination board recently appointed a committee to consider this question and a conference⁵ on this subject was held by representatives of the college entrance examination board, members of the national committee, and others. The following recommendations are taken from the report of the committee just referred to:

Fully one-third of the questions should be based on topics of such fundamental importance that they will have been thoroughly taught, carefully reviewed, and deeply impressed by effective drill. . . . They should be of such a degree of difficulty that any pupil of regular attendance, faithful application, and even moderate ability may be expected to answer them satisfactorily.

There should be both simple and difficult questions testing the candidate's ability to apply the principles of the subject. The early ones of the easy questions should be really easy for the candidate of good average ability who can do a little thinking under the stress of an examination; but even these questions should have genuine scientific content.

There should be a substantial question which will put the best candidates on their mettle, but which is not beyond the reach of a fair proportion of the really good candidates. This question should test the normal workings of a well-trained mind. It should be capable of being thought out in the limited time of the examination. It should be a test of the candidate's grasp and insight—not a catch question or a question of unfamiliar character making extraordinary demands on the critical powers of the candidate, or one the solution of which depends on an inspiration. Above all, this question should lie near to the heart of the subject as all well-prepared candidates understand the subject.

As a rule, a question should consist of a single part and be framed to test one thing—not pieced together out of several unrelated and perhaps unequally important parts.

Each question should be a substantial test on the topic or topics which it represents. It is, however, in the nature of the case impossible that all questions be of equal value.

Care should be used that the examination be not too long. * * * The examiner should be content to ask questions on the important topics, so chosen that their answers will be fair to the candidate and instructive to the readers; and beyond this merely to sample the candidate's knowledge on the minor topics.

The national committee suggests the following additional principles: The examination as a whole should, as far as practicable, reflect the principle that algebraic technique is a means to an end, and not an end in itself.

Questions that require of the candidate skill in algebraic manipulation beyond the needs of actual application should be used very sparingly.

An effort should be made to devise questions which will fairly test the candidate's understanding of principles and his ability to apply them, while involving a minimum of manipulative complexity.

⁵ At this conference the following vote was unanimously passed: "Voted, that the results of examinations (of the college entrance examination board,) be reported by letters A, B, C, D, E and that the definition of the groups represented by these letters should be determined in each year by the distribution of ability in a standard group of papers representing widely both public and private schools."

The examinations in geometry should be definitely constructed to test the candidate's ability to draw valid conclusions rather than his ability to memorize an argument.

A chapter on mathematical terms and symbols is included in this report.⁶ It is hoped that examining bodies will be guided by the recommendations there made relative to the use of terms and symbols in elementary mathematics.

The college entrance examination board, early in 1921, appointed a commission to recommend such revisions as might seem necessary in the definitions of the requirements in the various subjects of elementary mathematics. The recommendations contained in the present chapter have been laid before this commission. It is hoped that the commission's report, when it is finally made effective by action of the college entrance examination board and the various colleges concerned, will give impetus to the reorganization of the teaching of elementary mathematics along the lines recommended in the report of the national committee.

⁶ See Ch. VIII.

Chapter VI.

LISTS OF PROPOSITIONS IN PLANE AND SOLID GEOMETRY.

General basis of the selection of material.—The subcommittee appointed to prepare a list of basal propositions made a careful study of a number of widely used textbooks on geometry. The bases of selection of the propositions were two: (1) The extent to which the propositions and corollaries were used in subsequent proofs of important propositions and exercises; (2) the value of the propositions in completing important pieces of theory. Although the list of theorems and problems is substantially the same in nearly all textbooks in general use in this country, the wording, the sequence, and the methods of proof vary to such an extent as to render difficult a definite statement as to the number of times a proposition is used in the several books examined. A tentative table showed, however, less variation than might have been anticipated.

Classification of propositions.—The classification of propositions is not the same in plane geometry as in solid geometry. This is partly due to the fact that it is generally felt that the student should limit his construction work to figures in a plane and in which the compasses and straight edge are sufficient. The propositions have been divided as follows:

Plane geometry: I. Assumptions and theorems for informal treatment; II. Fundamental theorems and constructions: A. Theorems, B. Constructions; III. Subsidiary theorems.

Solid geometry: I. Fundamental theorems; II. Fundamental propositions in mensuration; III. Subsidiary theorems; IV. Subsidiary propositions in mensuration.

PLANE GEOMETRY.

I. *Assumptions and theorems for informal treatment.*—This list contains propositions which may be assumed without proof (postulates), and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs in certain cases, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed. The list should be taken as representative of

the type of propositions which may be assumed, or treated informally, rather than as exhaustive.

1. Through two distinct points it is possible to draw one straight line, and only one.
2. A line segment may be produced to any desired length.
3. The shortest path between two points is the line segment joining them.
4. One and only one perpendicular can be drawn through a given point to a given straight line.
5. The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6. From a given center and with a given radius one and only one circle can be described in a plane.
7. A straight line intersects a circle in at most two points.
8. Any figure may be moved from one place to another without changing its shape or size.
9. All right angles are equal.
10. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.
11. Equal angles have equal complements and equal supplements.
12. Vertical angles are equal.
13. Two lines perpendicular to the same line are parallel.
14. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.
15. Two lines parallel to the same line are parallel to each other.
16. The area of a rectangle is equal to its base times its altitude.

II. Fundamental theorems and constructions.—It is recommended that theorems and constructions (other than originals) to be proved on college entrance examinations be chosen from the following list. Originals and other exercises should be capable of solution by direct reference to one or more of these propositions and constructions. It should be obvious that any course in geometry that is capable of giving adequate training must include considerable additional material. The order here given is not intended to signify anything as to the order of presentation. It should be clearly understood that certain of the statements contain two or more theorems, and that the precise wording is not essential. The committee favors entire freedom in statement and sequence.

A. THEOREMS.

1. Two triangles are congruent if¹ (a) two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other; (b) two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other; (c) the three sides of one are equal, respectively, to the three sides of the other.
2. Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.
3. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.²
4. The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

¹ Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example, in 1, "Two triangles are equal if" * * * "a triangle is determined by" * * *, etc. Similarly in 2, the statement might read: "Two right triangles are congruent if, beside the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."

² It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore a converse may occasionally be accepted as true in an elementary course, if the necessity for proof is made clear. The proof may then be given later.

5. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.
6. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely.
7. The sum of the angles of a triangle is two right angles.
8. A parallelogram is divided into congruent triangles by either diagonal.
9. Any (convex) quadrilateral is a parallelogram (a) if the opposite sides are equal; (b) if two sides are equal and parallel.
10. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.
11. (a) The area of a parallelogram is equal to the base times the altitude.
(b) The area of a triangle is equal to one-half the base times the altitude.
(c) The area of a trapezoid is equal to half the sum of its bases times its altitude.
(d) The area of a regular polygon is equal to half the product of its apothem and perimeter.
12. (a) If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally.
(b) If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proofs for commensurable cases only.)
(c) The segments cut off on two transversals by a series of parallels are proportional.
13. Two triangles are similar if (a) they have two angles of one equal, respectively, to two angles of the other; (b) they have an angle of one equal to an angle of the other and the including sides are proportional; (c) their sides are respectively proportional.
14. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.
15. The perimeters of two similar polygons have the same ratio as any two corresponding sides.
16. Polygons are similar, if they can be decomposed into triangles which are similar and similarly placed; and conversely.
17. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.
18. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.
19. In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle.
20. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.
21. In the same circle, or in equal circles, if two arcs are equal, their central angles are equal; and conversely.
22. In any circle angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)
23. In the same circle or in equal circles, if two chords are equal their corresponding arcs are equal; and conversely.
24. (a) A diameter perpendicular to a chord bisects the chord and the arcs of the chord. (b) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.
25. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.
26. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.
27. An angle inscribed in a circle is equal to half the central angle having the same arc.
28. Angles inscribed in the same segment are equal.
29. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents at the points of division form a regular circumscribed polygon.
30. The circumference of a circle is equal to $2\pi r$. (Informal proof only.)
- 31.³ The area of a circle is equal to πr^2 . (Informal proof only.)

The treatment of the mensuration of the circle should be based upon related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits.

³ The total number of theorems given in this list when separated, as will probably be found advantageous in teaching this number including the converses indicated, is 52.

B. CONSTRUCTIONS.

1. Bisect a line segment and draw the perpendicular bisector.
2. Bisect an angle.
3. Construct a perpendicular to a given line through a given point.
4. Construct an angle equal to a given angle.
5. Through a given point draw a straight line parallel to a given straight line.
6. Construct a triangle, given (a) the three sides; (b) two sides and the included angle; (c) two angles and the included side.
7. Divide a line segment into parts proportional to given segments.
8. Given an arc of a circle, find its center.
9. Circumscribe a circle about a triangle.
10. Inscribe a circle in a triangle.
11. Construct a tangent to a circle through a given point.
12. Construct the fourth proportional to three given line segments.
13. Construct the mean proportional between two given line segments.
14. Construct a triangle (polygon) similar to a given triangle (polygon).
15. Construct a triangle equal to a given polygon.
16. Inscribe a square in a circle.
17. Inscribe a regular hexagon in a circle.

III. Subsidiary list of propositions.—The following list of propositions is intended to suggest some of the additional material referred to in the introductory paragraph of Section II. It is not intended, however, to be exhaustive; indeed, the committee feels that teachers should be allowed considerable freedom in the selection of such additional material, theorems, corollaries, originals, exercises, etc., in the hope that opportunity will thus be afforded for constructive work in the development of courses in geometry.

1. When two lines are cut by a transversal, if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.
2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.
3. A line perpendicular to one of two parallels is perpendicular to the other also.
4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.
5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.
6. The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.
7. In any parallelogram (a) the opposite sides are equal; (b) the opposite angles are equal; (c) the diagonals bisect each other.
8. Any (convex) quadrilateral is a parallelogram, if (a) the opposite angles are equal; (b) the diagonals bisect each other.
9. The medians of a triangle intersect in a point which is two-thirds of the distance from the vertex to the mid-point of the opposite side.
10. The altitudes of a triangle meet in a point.
11. The perpendicular bisectors of the sides of a triangle meet in a point.
12. The bisectors of the angles of a triangle meet in a point.
13. The tangents to a circle from an external point are equal.
- 14.^a (a) If two sides of a triangle are unequal, the greater side has the greater angle opposite it, and conversely.
(b) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second, and conversely.
(c) If two chords are unequal, the greater is at the less distance from the center, and conversely.

^a Such inequality theorems as these are of importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in the "Subsidiary list of propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

(d) The greater of two minor arcs has the greater chord, and conversely.
 15. An angle inscribed in a semicircle is a right angle.
 16. Parallel lines tangent to or cutting a circle intercept equal arcs on the circle.
 17. An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.
 18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.
 19. An angle formed by two secants or by two tangents to a circle is measured by half the difference between the intercepted arcs.
 20. If from a point without circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.
 21. Parallelograms or triangles of equal bases and altitudes are equal.
 22. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.

SOLID GEOMETRY.

In the following list the precise wording and the sequence are not considered:

I. FUNDAMENTAL THEOREMS.

1. If two planes meet, they intersect in a straight line.
2. If a line is perpendicular to each of two intersecting lines at their point of intersection it is perpendicular to the plane of the two lines.
3. Every perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point.
4. Through a given point (internal or external) there can pass one and only one perpendicular to a plane.
5. Two lines perpendicular to the same plane are parallel.
6. If two lines are parallel, every plane containing one of the lines and only one is parallel to the other.
7. Two planes perpendicular to the same line are parallel.
8. If two parallel planes are cut by a third plane, the lines of intersection are parallel.
9. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel.
10. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.
11. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.
12. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.
13. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.
14. An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.
15. The opposite faces of a parallelopiped are congruent.
16. The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equal triangular prisms.
17. If a pyramid or a cone is cut by a plane parallel to the base:
 - (a) The lateral edges and the altitude are divided proportionally;
 - (b) The section is a figure similar to the base;
 - (c) The area of the section is to the area of the base as the square of the distance from the vertex is to the square of the altitude of the pyramid or cone.
18. Two triangular pyramids having equal bases and equal altitudes are equal.
19. All points on a circle of a sphere are equidistant from either pole of the circle.
20. On any sphere a point which is at a quadrant's distance from each of two other points not the extremities of a diameter is a pole of the great circle passing through these two points.
21. If a plane is perpendicular to a radius at its extremity on a sphere, it is tangent to the sphere.
22. A sphere can be inscribed in or circumscribed about any tetrahedron.
23. If one spherical triangle is the polar of another, then reciprocally the second is the polar triangle of the first.
24. In two polar triangles each angle of either is the supplement of the opposite side of the other.
25. Two symmetric spherical triangles are equal.

II. FUNDAMENTAL PROPOSITIONS IN MENSURATION.

26. The lateral area of a prism or a circular cylinder is equal to the product of a lateral edge or element, respectively, by the perimeter of a right section.
27. The volume of a prism (including any parallelopiped) or of a circular cylinder is equal to the product of its base by its altitude.
28. The lateral area of a regular pyramid or a right circular cone is equal to half the product of its slant height by the perimeter of its base.
29. The volume of a pyramid or a cone is equal to one-third the product of its base by its altitude.
30. The area of a sphere.
31. The area of a spherical polygon.
32. The volume of a sphere.

III. SUBSIDIARY THEOREMS.

33. If from an external point a perpendicular and obliques are drawn to a plane. (a) the perpendicular is shorter than any oblique; (b) obliques meeting the plane at equal distances from the foot of the perpendicular are equal; (c) of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.
34. If two lines are cut by three parallel planes, their corresponding segments are proportional.
35. Between two lines not in the same plane there is one common perpendicular, and only one.
36. The bases of a cylinder are congruent.
37. If a plane intersects a sphere, the line of intersection is a circle.
38. The volume of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.
39. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.
40. Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.
41. The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.
42. The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.
43. If three face angles of one trihedral angle are equal, respectively, to the three face angles of another the trihedral angles are either congruent or symmetric.
44. Two spherical triangles on the same sphere are either congruent or symmetric if (a) two sides and the included angle of one are equal to the corresponding parts of the other; (b) two angles and the included side of one are equal to the corresponding parts of the other; (c) they are mutually equilateral; (d) they are mutually equangular.
45. The sum of any two face angles of a trihedral angle is greater than the third face angle.
46. The sum of the face angles of any convex polyhedral angle is less than four right angles.
47. Each side of a spherical triangle is less than the sum of the other two sides.
48. The sum of the sides of a spherical polygon is less than 360° .
49. The sum of the angles of a spherical triangle is greater than 180° and less than 540° .
50. There can not be more than five regular polyhedrons.
51. The locus of points equidistant (a) from two given points; (b) from two given planes which intersect.

IV. SUBSIDIARY PROPOSITIONS IN MENSURATION.

52. The volume of a frustum of (a) a pyramid or (b) a cone.
53. The lateral area of a frustum of (a) pyramid or (b) a cone of revolution.
54. The volume of a prismoid (without formal proof).

Chapter VII.

THE FUNCTION CONCEPT IN SECONDARY-SCHOOL MATHEMATICS.¹

In Chapter II, and incidentally in later chapters, considerable emphasis has been placed on the function concept or, better, on the idea of relationship between variable quantities as one of the general ideas that should dominate instruction in elementary mathematics. Since this recommendation is peculiarly open to misunderstanding on the part of teachers, it seems desirable to devote a separate chapter to a rather detailed discussion of what the recommendation means and implies.

It will be seen in what follows that there is no disposition to advocate the teaching of any sort of function *theory*. A prime danger of misconception that should be removed at the very outset is that teachers may think it is the notation and the definitions of such a theory that are to be taught. Nothing could be further from the intention of the committee. Indeed, it seems entirely safe to say that probably the word "function" had best not be used at all in the early courses.

What is desired is that the idea of relationship or dependence between variable quantities be imparted to the pupil by the examination of numerous concrete instances of such relationship. He must be shown the workings of relationships in a large number of cases before the abstract idea of relationship will have any meaning for him. Furthermore, the pupil should be led to form the habit of thinking about the connections that exist between related quantities, not merely because such a habit forms the best foundation for a real appreciation of the theory that may follow later, but chiefly because this habit will enable him to think more clearly about the quantities with which he will have to deal in real life, whether or not he takes any further work in mathematics.

Indeed, the reason for insisting so strongly upon attention to the idea of relationships between quantities is that such relationships do occur in real life in connection with practically all of the quantities with which we are called upon to deal in practice. Whereas there can be little doubt about the small value to the student who does

¹ The first draft of this chapter was prepared for the national committee by E. R. Hedrick, of the University of Missouri. It was discussed at the meeting of the committee, Sept. 2-4, 1920; revised by the author, and again discussed Dec. 29-30, 1920, and is now issued as part of the committee's report.

not go on to higher studies of some of the manipulative processes criticized by the national committee, there can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other.

To attain what has been suggested, the teacher should have in mind constantly not any definition to be recited by the pupil, not any automatic response to a given cue, not any memory exercise at all, but rather a determination not to pass any instance in which one quantity is related to another, or in which one quantity is determined by one or more others, without calling attention to the fact, and trying to have the student "see how it works." These instances occur in literally thousands of cases in both algebra and geometry. It is the purpose of this chapter to outline in some detail a few typical instances of this character.

RELATIONSHIPS IN ALGEBRA.

The instance of the function idea which usually occurs to one first in algebra is in connection with the study graphs. While this is natural enough, and while it is true that the graph is fundamentally functional in character, the supposition that it furnishes the first opportunity for observing functional relations between quantities betrays a misconception that ought to be corrected.

1. *Use of letters for numbers.*—The very first illustrations given in algebra to show the use of letters in the place of numbers are essentially functional in character. Thus, such relations as $I = prt$ and $A = \pi r^2$, as well as others that are frequently used, are statements of general relationships. These should be used to accustom the student not only to the use of letters in the place of numbers and to the solution of simple numerical problems, but also to the idea, for example, that changes in r affect the value of A . Such questions as the following should be considered: If r is doubled, what will happen to A ? If p is doubled, what will happen to I ? Appreciation of the meaning of such relationships will tend to clarify the entire subject under consideration. Without such an appreciation it may be doubted whether the student has any real grasp of the matter.

2. *Equations.*—Every simple problem leading to an equation in the first part of algebra would be better understood for just such a discussion as that mentioned above. Thus, if two dozen eggs are weighed in a basket which weighs 2 pounds, and if the total weight is found to be 5 pounds, what is the average weight of an egg? If x is the weight in ounces of one egg, the total weight with the 2-pound basket would be $24x + 32$ ounces. If the student will first try the effect of an average weight of 1 ounce, of $1\frac{1}{2}$ ounces, 2 ounces, $2\frac{1}{2}$ ounces, the meaning of the problem will stand out clearly. In

every such problem preliminary trials really amount to a discussion of the properties of a linear function.

3. *Formulas of pure science and of practical affairs.*—The study of formulas as such, aside from their numerical evaluation, is becoming of more and more importance. The actual uses of algebra are not to be found solely nor even principally in the solution of numerical problems for numerical answers. In such formulas as those for falling bodies, levers, etc., the manner in which changes in one quantity cause (or correspond to) changes in another are of prime importance, and their discussion need cause no difficulty whatever. The formulas under discussion here include those formulas of pure science and of practical affairs which are being introduced more and more into our texts on algebra. Whenever such a formula is encountered the teacher should be sure that the students have some comprehension of the effects of changes in one of the quantities upon the other quantity or quantities in the formula.

As a specific instance of such scientific formulas consider, for example, the force F , in pounds, with which a weight W , in pounds, pulls outward on a string (centrifugal force) if the weight is revolved rapidly at a speed v , in feet per second, at the end of a string of length r feet. This force is given by the formula $F = \frac{Wv^2}{32r}$. When such a formula is used the teacher should not be contented with the mere insertion of numerical values for W , v , and r to obtain a numerical value for F .

The advantage obtained from the study of such a formula lies quite as much in the recognition of the behavior of the force when one of the other quantities varies. Thus the student should be able to answer intelligently such questions as the following: If the weight is assumed to be twice as heavy, what is the effect upon the force? If the speed is taken twice as great, what is the effect upon the force? If the radius becomes twice as large, what is the effect upon the force? If the speed is doubled, what change in the weight would result in the same force? Will an increase in the speed cause an increase or a decrease in the force? Will an increase in the radius r cause an increase or a decrease in the force?

As another instance (of a more advanced character) consider the formula for the amount of a sum of money P , at compound interest at r per cent, at the end of n years. This amount may be denoted by A_n . Then we shall have $A_n = P(1+r)^n$. Will doubling P result in doubling A_n ? Will doubling n result in doubling A_n ? Since the compound interest that has accumulated is equal to the difference between P and A_n , will the doubling of r double the interest? Compare the correct answers to these questions with the answers to the similar questions in the case of simple interest, in

which the formula reads $A_n = P + Prn$ and in which the accumulated interest is simply Prn .

The difference between such a study of the effect produced upon one quantity by changes in another and the mere substitution of numerical values will be apparent from these examples.

4. *Formulas of pure algebra.*—Formulas of pure algebra, such as that for $(x+h)^2$, will be better understood and appreciated if accompanied by a discussion of the manner in which changes in h cause changes in the total result. This can be accomplished by discussing such concrete realities as the error made in computing the area of a square field or of a square room when an error has been made in measuring the side of the square. If x is the true length of the side, and if the student assumes various possible values for the error h made in measuring x , he will have a situation that involves some comprehension of the functional workings of the formula mentioned. The same formula relates to such problems as the change from one entry to the next entry in a table of squares.

A similar situation, and a very important one, occurs with the pure algebraic formula for $(x+a)(y+b)$. This formula may be said to govern the question of the keeping of significant figures in finding the product xy . For if a and b represent the uncertainty in x and y , respectively, the uncertainty in the product is given by this formula. The student has much to learn on this score, for the retention of meaningless figures in a product is one of the commonest mistakes of both student and teacher in computational work.

Such formulas occur throughout algebra, and each of them will be illuminated by such a discussion. The formulas for arithmetic and geometric progression, for example, should be studied from a functional standpoint.

5. *Tables.*—The uses of the functional idea in connection with numerical computation have already been mentioned in connection with the formula for a product. Work which appears on the surface to be wholly numerical may be of a distinctly functional character. Thus any table, e. g., a table of squares, corresponds to or is constructed from a functional relation, e. g., for a table of squares, the relation $y = x^2$. The differences in such a table are the differences caused by changes in the values of the independent variable. Thus, the differences in a table of squares are precisely the differences between x^2 and $(x+h)^2$ for various values of x .

6. *Graphs.*—The functional character of graphical representations was mentioned at the beginning of this section. Every graph is obviously a representation of a functional relationship between two or more quantities. What is needed is only to draw attention to this fact and to study each graph from this standpoint. In addition to this, however, it is desirable to point out that functional

relations may be studied directly by means of graphs without the intervention of any algebraic formula. Thus such a graph as a population curve, or a curve representing wind pressure, obviously represents a relationship between two quantities, but there is no known formula in either case. The idea that the three concepts, tables, graphs, algebraic formulas, are all representations of the same kind of connection between quantities, and that we may start in some instances with any of the three, is a most valuable addition to the student's mental equipment, and to his control over the quantities with which he will deal in his daily life.

RELATIONSHIPS IN GEOMETRY.

Thus far the instances mentioned have been largely algebraic, though certain mensuration formulas of geometry have been mentioned. While the mensuration formulas may occur to one first as an illustration of functional concepts in geometry, they are by no means the earliest relationships that occur in that study.

1. *Congruence*.—Among the earliest theorems are those on the congruence of triangles. In any such theorem, the parts necessary to establish congruence evidently determine completely the size of each other part. Thus, two sides and the included angle of a triangle evidently determine the length of the third side. If the student clearly grasps this fact, the meaning of this case of congruence will be more vivid to him, and he will be prepared for its important applications in surveying and in trigonometry. Even if he never studies those subjects, he will nevertheless be able to use his understanding of the situation in any practical cases in which the angle between two fixed rods or beams is to be fixed or is to be determined, in a practical situation such as house building. Other congruence theorems throughout geometry may well be treated in a similar manner.

2. *Inequalities*.—In the theorems regarding inequalities, the functional quality is even more pronounced. Thus, if two triangles have two sides of one equal respectively to two sides of the other, but if the included angle between these sides in the one triangle is greater than the corresponding angle in the other, then the third sides of the triangles are unequal in the same sense. This theorem shows that as one angle grows, the side opposite it grows, if the other sides remain unchanged. A full realization of the fact here mentioned would involve a real grasp of the functional relation between the angle and the side opposite it. Thus, if the angle is doubled, will the side opposite it be doubled? Such questions arise in connection with all theorems on inequalities.

3. *Variations in figures*.—A great assistance to the imagination is gained in certain figures by imagining variations of the figure through

all intermediate stages from one case to another. Thus, the angle between two lines that cut a circle is measured by a proper combination of the two arcs cut out of the circle by the two lines. As the vertex of the angle passes from the center of the circle to the circumference and thence to the outside of the circle, the rule changes, but these changes may be borne in mind, and the entire scheme may be grasped, by imagining a continuous change from the one position to the other, following all the time the changes in the intercepted arcs. The angle between a secant and a tangent is measured in a manner that can best be grasped by another such continuous motion, watching the changes in the measuring arcs as the motion occurs. Such observations are essentially functional in character, for they consist in careful observations of the relationships between the angle to be measured and the arcs that measure it.

4. *Motion*.—The preceding discussion of variable figures leads naturally to a discussion of actual motion. As figures move, either in whole or in part, the relationships between the quantities involved may change. To note these changes is to study the functional relationships between the parts of the figures. Without the functional idea, geometry would be wholly *static*. The study of fixed figures should not be the sole purpose of a course in geometry, for the uses of geometry are not wholly on static figures. Indeed, in all machinery, the geometric figures formed are in continual motion, and the shapes of the figures formed by the moving parts change. The study of motion and of moving forms, the *dynamic* aspects of geometry, should be given at least some consideration. Whenever this is done, the functional relations between the parts become of prime importance. Thus a linkage of the form of a parallelogram can be made more nearly rectangular by making the diagonals more nearly equal, and the linkage becomes a rectangle if the diagonals are made exactly equal. This principle is used in practice in making a rectangular framework precisely true.

5. *Proportionality theorems*.—All theorems which assert that certain quantities are in proportion to certain others, are obviously functional in character. Thus even the simplest theorems on rectangles assert that the area of a rectangle is directly proportional to its height, if the base is fixed. When more serious theorems are reached, such as the theorems on similar triangles, the functional ideas involved are worthy of considerable attention. That this is eminently true will be realized by all to whom trigonometry is familiar, for the trigonometric functions are nothing but the ratios of the sides of right triangles. But even in the field of elementary geometry a clear understanding of the relation between the areas (and volumes) of similar figures and the corresponding linear dimensions is of prime importance.

RELATIONSHIPS IN TRIGONOMETRY.

The existence of functional relationships in trigonometry is evidenced by the common use of the words "trigonometric functions" to describe the trigonometric ratios. Thus the sine of an angle is a definite ratio, whose value depends upon and is determined by the size of the angle to which it refers. The student should be made conscious of this relationship and he should be asked such questions as the following: Does the sine of an angle increase or decrease as the angle changes from zero to 90° ? If the angle is doubled, does the sine of the angle double? If not, is the sine of double the angle more or less than twice the sine of the original angle? How does the value of the sine behave as the angle increases from 90 to 180° ? From 180 to 270° ? From 270 to 360° ? Similar questions may be asked for the cosine and for the tangent of an angle.

Such questions may be reinforced by the use of figures that illustrate the points in question. Thus an angle twice a given angle should be drawn, and its sine should be estimated from the figure. A central angle and an inscribed angle on the same arc may be drawn in any circle. If they have one side in common, the relations between their sines will be more apparent. Finally the relationships that exist may be made vivid by actual comparison of the numerical values found from the trigonometric tables.

Not only in these first functional definitions, however, but in a variety of geometric figures throughout trigonometry do functional relations appear. Thus the law of cosines states a definite relationship between the three sides of a triangle and any one of the angles. How will the angle be affected by increase or decrease of the side opposite it, if the other two sides remain fixed? How will the angle be affected by an increase or a decrease of one of the adjacent sides, if the other two sides remain fixed? Are these statements still true if the angle in question is obtuse?

As another example, the height of a tree, or the height of a building, may be determined by measuring the two angles of elevation from two points on the level plain in a straight line with its base. A formula for the height (h) in terms of these two angles (A, B) and the distance (d) between the points of observation, may be easily written down ($h = d \sin A \sin B / \sin (A - B)$). Then the effect upon the height of changes in one of these angles may be discussed.

In a similar manner, every formula that is given or derived in a course on trigonometry may be discussed with profit from the functional standpoint.

CONCLUSION.

In conclusion, mention should be made of the great rôle which the idea of functions plays in the life of the world about us. Even when no calculation is to be carried out, the problems of real life frequently

involve the ability to think correctly about the nature of the relationships which exist between related quantities. Specific mention has been made already of this type of problem in connection with interest on money. In everyday affairs, such as the filling out of formulas for fertilizers or for feeds, or for spraying mixtures on the farm, the similar filling out of recipes for cooking (on different scales from that of the book of recipes), or the proper balancing of the ration in the preparation of food, many persons are at a loss on account of their lack of training in thinking about the relations between quantities. Another such instance of very common occurrence in real life is in insurance. Very few men or women attempt intelligently to understand the meaning and the fairness of premiums on life insurance and on other forms of insurance, chiefly because they can not readily grasp the relations of interest and of chance that are involved. These relations are not particularly complicated and they do not involve any great amount of calculation for the comprehension of the meaning and of the fairness of the rates. Mechanics, farmers, merchants, housewives, as well as scientists, and engineers have to do constantly with quantities of things, and the quantities with which they deal are related to other quantities in ways that require clear thinking for maximum efficiency.

One element that should not be neglected is the occurrence of such problems in public questions which must be decided by the votes of the whole people. The tariff, rates of postage and express, freight rates, regulation of insurance rates, income taxes, inheritance taxes, and many other public questions involve relationships between quantities—for example, between the rate of income taxation and the amount of the income—that require habits of functional thinking for intelligent decisions. The training in such habits of thinking is therefore a vital element toward the creation of good citizenship.

It is believed that transfer of training does operate between such topics as those suggested in the body of this paper and those just mentioned, because of the existence of such identical or common elements, whereas the transfer of the training given by courses in mathematics that do not emphasize functional relationships might be questionable.

While this account of the functional character of certain topics in geometry and in algebra makes no claim to being exhaustive, the topics mentioned will suggest others of like character to the thoughtful teacher. It is hoped that sufficient variety has been mentioned to demonstrate the existence of functional ideas throughout elementary algebra and geometry. The committee feels that if this is recognized, algebra and geometry can be given new meaning to many children, and that all students will be better able to control the actual relations which they meet in their own lives.

Chapter VIII.

TERMS AND SYMBOLS IN ELEMENTARY MATHEMATICS.¹

A. Limitations imposed by the committee upon its work.—The committee feels that in dealing with this subject it should explicitly recognize certain general limitations, as follows:

1. No attempt should be made to impose the phraseology of any definition, although the committee should state clearly its general views as to the meaning of disputed terms.

2. No effort should be made to change any well-defined current usage unless there is a strong reason for doing so, which reason is supported by the best authority, and, other things being substantially equal, the terms used should be international. This principle excludes the use of all individual efforts at coining new terms except under circumstances of great urgency. The individual opinions of the members, as indeed of any teacher or body of teachers, should have little weight in comparison with general usage if this usage is definite. If an idea has to be expressed so often in elementary mathematics that it becomes necessary to invent a single term or symbol for the purpose, this invention is necessarily the work of an individual; but it is highly desirable, even in this case, that it should receive the sanction of wide use before it is adopted in any system of examinations.

3. On account of the large number of terms and symbols now in use, the recommendations to be made will necessarily be typical rather than exhaustive.

I. GEOMETRY.

B. Undefined terms.—The committee recommends that no attempt be made to define, with any approach to precision, terms whose definitions are not needed as parts of a proof.

Especially is it recommended that no attempt be made to define precisely such terms as *space*, *magnitude*, *point*, *straight line*, *surface*, *plane*, *direction*, *distance*, and *solid*, although the significance of such terms should be made clear by informal explanations and discussions.

C. Definite usage recommended.—It is the opinion of the committee that the following general usage is desirable:

1. *Circle* should be considered as the curve; but where no ambiguity arises, the word “circle” may be used to refer either to the curve or to the part of the plane inclosed by it.

¹ The first draft of this chapter was prepared by a subcommittee consisting of David Eugene Smith (chairman), W. W. Hart, H. E. Hawkes, E. R. Hedrick, and H. E. Slaught. It was revised by the national committee at its meeting December 29 and 30, 1920.

2. *Polygon* (including *triangle*, *square*, *parallelogram*, and the like) should be considered, by analogy to a circle, as a closed broken line; but where no ambiguity arises, the word polygon may be used to refer either to the broken line or to the part of the plane inclosed by it. Similarly, *segment of a circle* should be defined as the figure formed by a chord and either of its arcs.

3. *Area of a circle* should be defined as the area (numerical measure) of the portion of the plane inclosed by the circle. *Area of a polygon* should be treated in the same way.

4. *Solids*. The usage above recommended with respect to plane figures is also recommended with respect to solids. For example, *sphere* should be regarded as a surface, its volume should be defined in a manner similar to the area of a circle, and the double use of the word should be allowed where no ambiguity arises. A similar usage should obtain with respect to such terms as *polyhedron*, *cone*, and *cylinder*.

5. *Circumference* should be considered as the length (numerical measure) of the circle (line). Similarly, *perimeter* should be defined as the length of the broken line which forms a polygon; that is, as the sum of the lengths of the sides.

6. *Obtuse angle* should be defined as an angle greater than a right angle and less than a straight angle, and should therefore not be defined merely as an angle greater than a right angle.

7. The term *right triangle* should be preferred to "right-angled triangle," this usage being now so standardized in this country that it may properly be continued in spite of the fact that it is not international. Similarly for *acute triangle*, *obtuse triangle*, and *oblique triangle*.

8. Such English plurals as *formulas* and *polyhedrons* should be used in place of the Latin and Greek plurals. Such unnecessary Latin abbreviations as *Q. E. D.* and *Q. E. F.* should be dropped.

9. The definitions of *axiom* and *postulate* vary so much that the committee does not undertake to distinguish between them.

D. *Terms made general*.—It is the recommendation of the committee that the modern tendency of having terms made as general as possible should be followed. For example:

1. *Isosceles triangle* should be defined as a triangle having two equal sides. There should be no limitation to two and only two equal sides.

2. *Rectangle* should be considered as including a square as a special case.

3. *Parallelogram* should be considered as including a rectangle, and hence a square, as a special case.

4. *Segment* should be used to designate the part of a straight line included between two of its points as well as the figure formed by an

arc of a circle and its chord, this being the usage generally recognized by modern writers.

E. Terms to be abandoned.—It is the opinion of the committee that the following terms are not of enough consequence in elementary mathematics at the present time to make their recognition desirable in examinations, and that they serve chiefly to increase the technical vocabulary to the point of being burdensome and unnecessary:

1. *Antecedent and consequent.*
2. *Third proportional and fourth proportional.*
3. *Equivalent.* An unnecessary substitute for the more precise expressions "equal in area" and "equal in volume," or (where no confusion is likely to arise) for the single word "equal."
4. *Trapezium.*
5. *Scholium, lemma, oblong, scalene triangle, sect, perigon, rhomboid* (the term "oblique parallelogram" being sufficient), and *reflex angle* (in elementary geometry).
6. Terms like *flat angle, whole angle, and conjugate angle* are not of enough value in an elementary course to make it desirable to recommend them.
7. *Subtend*, a word which has no longer any etymological meaning to most students and teachers of geometry. While its use will naturally continue for some time to come, teachers may safely incline to such forms as the following: "In the same circle equal arcs have equal chords."
8. *Homologous*, the less technical term "corresponding" being preferable.
9. Guided by principle A2 and its interpretation, the committee advises against the use of such terms as the following: *Angle-bisector, angle-sum, consecutive interior angles, supplementary consecutive exterior angles, quader* (for rectangular solid), *sect, explement, transverse angles.*
10. It is unfortunate that it still seems to be necessary to use such a term as *parallelepiped*, but we seem to have no satisfactory substitute. For rectangular parallelepiped, however, the use of *rectangular solid* is recommended. If the terms were more generally used in elementary geometry it would be desirable to consider carefully whether better ones could not be found for the purposes than *isoperimetric, apothem, icosahedron, and dodecahedron.*

F. Symbols in elementary geometry.—It should be recognized that a symbol like \perp is merely a piece of shorthand designed to afford an easy grasp of a written or printed statement. Many teachers and a few writers make an extreme use of symbols, coining new ones to meet their own views as to usefulness, and this practice is

naturally open to objection.² There are, however, certain symbols that are international and certain others of which the meaning is at once apparent and which are sufficiently useful and generally enough recognized to be recommended.

For example, the symbols for triangle, \triangle , and circle, \odot , are international, although used more extensively in the United States than in other countries. Their use, with their customary plurals, is recommended.

The symbol \perp , generally read as representing the single word "perpendicular" but sometimes as standing for the phrase "is perpendicular to," is fairly international and the meaning is apparent. Its use is therefore recommended. On account of such a phrase as "the $\perp AB$," the first of the above readings is likely to be the more widely used, but in either case there is no chance for confusion.

The symbol \parallel for "parallel" or "is parallel to" is fairly international and is recommended.

The symbol \sim for "similar" or "is similar to" is international and is recommended.

The symbols \cong and \equiv for "congruent" or "is congruent to" both have a considerable use in this country. The committee feels that the former, which is fairly international, is to be preferred because it is the more distinctive and suggestive.

The symbol \angle for "angle" is, because of its simplicity, coming to be generally preferred to any other and is therefore recommended.

Since the following terms are not used frequently enough to render special symbols of any particular value, the world has not developed any that have general acceptance, and there seems to be no necessity for making the attempt: Square, rectangle, parallelogram, trapezoid, quadrilateral, semicircle.

The symbol \widehat{AB} for "arc AB " can not be called international. While the value of the symbol \smile in place of the short word *arc* is doubtful, the committee sees no objection to its use.

The symbol \therefore for "therefore" has a value that is generally recognized, but the symbol \because for "since" is used so seldom that it should be abandoned.

With respect to the lettering of figures, the committee calls attention for purposes of general information to a convenient method, found in certain European and in some American textbooks, of lettering triangles: Capitals represent the vertices, corresponding small letters represent opposite sides, corresponding small Greek letters represent angles, and the primed letters represent the corresponding parts of a congruent or similar triangle. This permits of

² This is not intended to discourage the use of algebraic methods in the solution of such geometric problems as lend themselves readily to algebraic treatment.

speaking of α (alpha) instead of "angle A," and of "small a " instead of BC . The plan is by no means international with respect to the Greek letters. The committee is prepared, however, to recommend it with the optional use of the Greek forms.

In general, it is recommended that a single letter be used to designate any geometric magnitude, whenever there is no danger of ambiguity. The use of numbers alone to designate magnitudes should be avoided by the use of such forms A_1, A_2, \dots .

With respect to the symbolism for limits, the committee calls attention to the fact that the symbol \doteq is a local one, and that the symbol \rightarrow (for "tends to") is both international and expressive and has constantly grown in favor in recent years. Although the subject of limits is not generally treated scientifically in the secondary school, the idea is mentioned in geometry and a symbol may occasionally be needed.

While the teacher should be allowed freedom in the matter, the committee feels that it is desirable to discourage the use of such purely local symbols as the following:

$\hat{=}$ for "equal in degrees,"

ass for "two sides and an angle adjacent to one of them," and
 sas for "two sides and the included angle."

G. Terms not standardized.—At the present time there is not sufficient agreement upon which to base recommendations as to the use of the term *ray* and as to the value of terms like *coplanar*, *collinear*, and *concurrent* in elementary work. Many terms, similar to these, will gradually become standardized or else will naturally drop out of use.

II. ALGEBRA AND ARITHMETIC.

H. Terms in algebra.—1. With respect to equations the committee calls attention to the fact that the classification according to degree is comparatively recent and that this probably accounts for the fact that the terminology is so unsettled. The Anglo-American custom of designating an equation of the first degree as a *simple equation* has never been satisfactory, because the term has no real significance. The most nearly international terms are *equation of the first degree* (or "first degree equation") and *linear equation*. The latter is so brief and suggestive that it should be generally adopted.

2. The term *quadratic equation* (for which the longer term "second degree equation" is an unnecessary synonym, although occasionally a convenient one) is well established. The terms *pure quadratic* and *affected quadratic* signify nothing to the pupil except as he learns the meaning from a book, and the committee recommends that they be dropped. Terms more nearly in general use are *complete quadratic* and *incomplete quadratic*. The committee feels, however, that the

distinction thus denoted is not of much importance and believes that it can well be dispensed with in elementary instruction.

3. As to other special terms, the committee recommends abandoning, so far as possible, the use of the following: *Aggregation* for grouping; *vinculum* for bar; *evolution* for finding roots, as a general topic; *involution* for finding powers; *extract* for find (a root); *absolute term* for constant term; *multiply an equation, clear of fractions, cancel* and *transpose*, at least until the significance of the terms is entirely clear; *aliquot part* (except in commercial work).

4. The committee also advises the use of either *system of equations* or *set of equations* instead of "simultaneous equations," in such an expression as "solve the following set of equations," in view of the fact that at present no well established definite meaning attaches to the term "simultaneous."

5. The term *simplify* should not be used in cases where there is possibility of misunderstanding. For purposes of computation, for example, the form $\sqrt{8}$ may be simpler than the form $2\sqrt{2}$, and in some cases it may be better to express $\sqrt{\frac{3}{4}}$ as $\sqrt{0.75}$ instead of $\frac{1}{2}\sqrt{3}$. In such cases, it is better to give more explicit instructions than to use the misleading term "simplify."

6. The committee regrets the general uncertainty in the use of the word *surd*, but it sees no reasonable chance at present of replacing it by a more definite term. It recognizes the difficulty generally met by young pupils in distinguishing between *coefficient* and *exponent*, but it feels that it is undesirable to attempt to change terms which have come to have a standardized meaning and which are reasonably simple. These considerations will probably lead to the retention of such terms as *rationalize*, *extraneous root*, *characteristic*, and *mantissa*, although in the case of the last two terms "integral part" and "fractional part" (of a logarithm) would seem to be desirable substitutes.

7. While recognizing the motive that has prompted a few teachers to speak of "positive x " instead of "plus x ," and "negative y " instead of "minus y ," the committee feels that attempts to change general usage should not be made when based upon trivial grounds and when not sanctioned by mathematicians generally.

1. *Symbols in algebra*.—The symbols in elementary algebra are now so well standardized as to require but few comments in a report of this kind. The committee feels that it is desirable, however, to call attention to the following details:

1. Owing to the frequent use of the letter x , it is preferable to use the center dot (a raised period) for multiplication in the few cases in which any symbol is necessary. For example, in a case like $1 \cdot 2 \cdot 3 \cdots (x - 1) \cdot x$, the center dot is preferable to the symbol \times ; but

in cases like $2a(x-a)$ no symbol is necessary. The committee recognizes that the period (as in $a.b$) is more nearly international than the center dot (as in $a \cdot b$); but inasmuch as the period will continue to be used in this country as a decimal point, it is likely to cause confusion, to elementary pupils at least, to attempt to use it as a symbol for multiplication.

2. With respect to division, the symbol \div is purely Anglo-American, the symbol : serving in most countries for division as well as ratio.

- Since neither symbol plays any part in business life, it seems proper to consider only the needs of algebra, and to make more use of the fractional form and (where the meaning is clear) of the symbol $/$, and to drop the symbol \div in writing algebraic expressions.

3. With respect to the distinction between the use of $+$ and $-$ as symbols of operation and as symbols of direction, the committee sees no reason for attempting to use smaller signs for the latter purpose, such an attempt never having received international recognition, and the need of two sets of symbols not being sufficient to warrant violating international usage and burdening the pupil with this additional symbolism.

4. With respect to the distinction between the symbols \equiv and $=$ as representing respectively identity and equality, the committee calls attention to the fact that, while the distinction is generally recognized, the consistent use of the symbols is rarely seen in practice. The committee recommends that the symbol \equiv be not employed in examinations for the purpose of indicating identity. The teacher, however, should use both symbols if desired.

5. With respect to the root sign, $\sqrt{}$, the committee recognizes that convenience of writing assures its continued use in many cases instead of the fractional exponent. It is recommended, however, that in algebraic work involving complicated cases the fractional exponent be preferred. Attention is also called to the fact that the convention is quite generally accepted that the symbol \sqrt{a} (a representing a positive number) means only the positive square root and that the symbol $\sqrt[n]{a}$ means only the principal n th root, and similarly for $a^{\frac{1}{2}}$, $a^{\frac{1}{n}}$. The reason for this convention is apparent when we come to consider the value of $\sqrt{4} = \sqrt{9} = \sqrt{16} = \sqrt{25}$. This convention being agreed to, it is improper to write $x = \sqrt{4}$ as the complete solution of $x^2 - 4 = 0$, but the result should appear as $x = \pm \sqrt{4}$. Similarly, it is not in accord with the convention to write $\sqrt{4} = \pm 2$, the conventional form being $\pm \sqrt{4} = \pm 2$; and for the same reason it is impossible to have $\sqrt{(-1)^2} = -1$, since the symbol refers only to a positive root. These distinctions are not matters to be settled by the individual opinion of a teacher or a local group of

teachers; they are purely matters of convention as to notation, and the committee simply sets forth, for the benefit of the teachers, this statement as to what the convention seems to be among the leading writers of the world at the present time.

When imaginaries are used, the symbol i should be employed instead of $\sqrt{-1}$ except possibly in the first presentation of the subject.

7. As to the factorial symbols $5!$ and $\underline{5}$, to represent $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, the tendency is very general to abandon the second one, probably on account of the difficulty of printing it, and the committee so recommends. This question is not, however, of much importance in the general courses in the high school.

8. With respect to symbols for an unknown quantity there has been a noteworthy change within a few years. While the Cartesian use of x and y will doubtless continue for two general unknowns, the recognition that the formula is, in the broad use of the term, a central feature of algebra has led to the extended use of the initial letter. This is simply illustrated in the direction to solve for r the equation $A = \pi r^2$. This custom is now international and should be fully recognized in the schools.

9. The committee advises abandoning the double colon (:) in proportion, and the symbol \propto in variation, both of these symbols being now practically obsolete.

J. Terms and symbols in arithmetic.—1. While it is rarely wise to attempt to abandon suddenly the use of words that are well established in our language, the committee feels called upon to express regret that we still require very young pupils, often in the primary grades, to use such terms as *subtrahend*, *addend*, *minuend* and *multiplicand*. Teachers themselves rarely understand the real significance of these words, nor do they recognize that they are comparatively modern additions to what used to be a much simpler vocabulary in arithmetic. The committee recommends that such terms be used, if at all, only after the sixth grade.

2. Owing to the uncertainty attached to such expressions as "to three decimal places," "to thousandths," "correct to three decimal places," "correct to the nearest thousandth," the following usage is recommended: When used to specify accuracy in computation, the four expressions should be regarded as identical. The expression "to three decimal places" or "to thousandths" may be used in giving directions as to the extent of a computation. It then refers to a result carried only to thousandths, without considering the figure of ten-thousandths; but it should be avoided as far as possible because it is open to misunderstanding. As to the term, "significant figure," it should be noted that 0 is always significant

except when used before a decimal fraction to indicate the absence of integers or, in general, when used merely to locate the decimal point. For example, the zeros underscored in the following are "significant," while the others are not: 0.5, 9.50, 102, 30,200. Further, the number 2396, if expressed correct to three significant figures, would be written 2400.³ It should be noted that the context or the way in which a number has been obtained is sometimes the determining factor as to the significance of a 0.

3. The pupil in arithmetic needs to see the work in the form in which he will use it in practical life outside the schoolroom. His visualization of the process should therefore not include such symbols as +, -, ×, ÷, which are helpful only in writing out the analysis of a problem or in the printed statement of the operation to be performed. Because of these facts the committee recommends that only slight use be made of these symbols in the written work of the pupil, except in the analysis of problems. It recognizes, however, the value of such symbols in printed directions and in these analyses.

III. GENERAL OBSERVATIONS AND RECOMMENDATIONS.

K. General observations.—The committee desires also to record its belief in two or three general observations.

1. It is very desirable to bring mathematical writing into closer touch with good usage in English writing in general. That we have failed in this particular has been the subject of frequent comment by teachers of mathematics as well as by teachers of English. This is all the more unfortunate because mathematics may be and should be a genuine help toward the acquisition of good habits in the speaking and writing of English. Under present conditions, with a style that is often stilted and in which undue compression is evident, we do not offer to the student the good models of English writing of which mathematics is capable, nor indeed do we always offer good models of thought processes. It is to be feared that many teachers encourage the use of a kind of vulgar mathematical slang when they allow such words as "tan" and "cos," for tangent and cosine, and habitually call their subject by the title "math."

2. In the same general spirit the committee wishes to observe that teachers of mathematics and writers of textbooks seem often to have gone to an extreme in searching for technical terms and for new symbols. The committee expresses the hope that mathematics may retain, as far as possible, a literary flavor. It seems perfectly feasible that a printed discussion should strike the pupil as an expression of reasonable ideas in terms of reasonable English forms. The fewer technical terms we introduce, the less is the subject likely to give

³ The underscoring of significant zeros is here used merely to make clear the committee's meaning. The device is not recommended for general adoption.

the impression of being difficult and a mere juggling of words and symbols.

3. While recognizing the claims of euphony, the fact that a word like "historic" has a different meaning from "historical" and that confusion may occasionally arise if "arithmetic" is used as an adjective with a different pronunciation from the noun, the committee advises that such forms as geometric be preferred to geometrical. This is already done in such terms as analytic geometry and elliptic functions, and it seems proper to extend the custom to include *arithmetic, geometric, graphic*, and the like.

L. General recommendations. —In view of the desirability of a simplification of terms used in elementary instruction, and of establishing international usage so far as is reasonable, the committee recommends that the subject of this report be considered by a committee to be appointed by Section IV of the next International Congress of Mathematicians, such committee to contain representatives of at least the recognized international languages admitted to the meetings.

2. The committee suggests that examining bodies, contributors to mathematical journals, and authors of textbooks endeavor to follow the general principles formulated in this report.

SYNOPSIS OF THE REMAINING CHAPTERS OF THE COMPLETE REPORT.

CHAPTER IX.—THE PRESENT STATUS OF DISCIPLINARY VALUES IN EDUCATION. By Vevia Blair.

A critical survey of the scientific literature relating to disciplinary values and the transfer of training, followed by an attempt to formulate conclusions warranted by the results of investigations and supported by leading educational psychologists.

CHAPTER X.—A CRITICAL STUDY OF THE CORRELATION METHOD APPLIED TO GRADES. By A. R. Crathorne.

An investigation to determine the serviceability and reliability in educational problems of the theory of correlation applied to grades. The chapter begins with a nontechnical description of the theory of correlation. This is followed by an attempt to formulate a scale of values for the correlation coefficient with a view of determining with as much precision as possible what constitutes "high," "medium," or "low" correlation. Finally the theory is applied to all the studies in the high-school curriculum—with some significant results as to the educational influence of certain groups of subjects as compared with other groups.

CHAPTER XI.—MATHEMATICAL CURRICULA IN FOREIGN COUNTRIES. By J. C. Brown.

A brief account of the work in mathematics offered by the elementary and secondary schools of Austria, Belgium, Denmark, France, Germany, Holland, Hungary, Italy, Japan, Roumania, Russia, Sweden, and Switzerland, followed by a general summary comparing conditions in the United States with those in the countries mentioned.

CHAPTER XII.—EXPERIMENTAL COURSES IN MATHEMATICS. By Raleigh Schorling.

An extensive description of the work in mathematics given in 14 leading experimental schools of this country—giving in detail the order of topics and the time devoted to each, the equipment of the schools, the cost of the instruction, etc.

CHAPTER XIII.—STANDARDIZED TESTS IN MATHEMATICS FOR SECONDARY SCHOOLS. By C. B. Upton.

A description of various standard tests in arithmetic, algebra, and geometry, and a discussion of their use in the teaching of mathematics. The tests considered are the following: The Courtis arithmetic tests, the Woody arithmetic scales, the Woody-McCall arithmetic test, the Stone reasoning tests, the Courtis standard practice tests in arithmetic, the Studebaker practice exercises in arithmetic, the Rugg-Clark practice exercises in first-year algebra, the Hotz first-year algebra scales, the Kelly mathematical values test, the Minnick geometry tests, the Rogers prognostic tests of mathematical ability, and others.

CHAPTER XIV. -THE TRAINING OF TEACHERS OF MATHEMATICS. By R. C. Archibald.

A selection from the results of an extended and detailed investigation, covering every State in the Union and all the larger cities, concerning present conditions regarding the training of teachers of mathematics, the facilities for providing such training, and the requirements for certification. The selection is made for the purpose of exhibiting the highest standards to be found in various parts of the country and the courses of study for the training of teachers given in various institutions. The chapter also includes a brief survey of conditions in certain foreign countries.

CHAPTER XV. -CERTAIN QUESTIONNAIRE INVESTIGATIONS. By W. F. Downey, A. R. Crathorne, Alfred Davis, and others.

These investigations relate to the interests of high-school pupils, change of mind as to life work between high school and college, the importance of mathematics, etc.

CHAPTER XVI.—BIBLIOGRAPHY ON THE TEACHING OF MATHEMATICS. By D. E. Smith and J. A. Foberg.

This bibliography lists all the articles relating to mathematics that have appeared since 1910 in a number of leading educational periodicals, and gives in addition to author, title, and place of publication a brief summary of each article.

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